Using Light Bulbs, Hats, Hercules, and the Hydra to Build Problem Solving Skills in Students

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Abstract

Many American classrooms have adopted a strategy which mostly focuses on rote memorization to teach math to its students. While this may sometimes help students pass multiple choice exams, we argue that this focus will not help students develop problem solving and critical thinking abilities. We created different activities targeted towards a wide range of age groups to see how students would respond to difficult problems that require perseverance and critical thinking. Each activity fit in with the Common Core Standards laid out by the Colorado Department of Education. Different activities catered to different age groups, but all of them were meant to be challenging for the audience. The culminating activity was shaped around a classic problem of Hercules fighting the Hydra. Hercules cuts off a head of the multi-headed dragon, and a certain amount of heads grow back based on rules we will define. We asked the students how the Hydra would behave as Hercules continued to attack. To help them with this problem, we compared it to a special group of fast growing sequences called Goodstein Sequences.

1 Introduction

Some people, me being one of them, do not agree with way that Mathematics is being taught in primary and secondary schools. American classrooms have adopted a strategy which mostly focuses on rote memorization for most of its students. This is due to a focus on standardized testing, meant to hold schools and teachers accountable. However, studies show that problems that these standardized test questions, which are taught with memorization and drill-and-kill mentality require no higher level thinking [3]. This means that students can solve these problems without developing any critical thinking skills. Unfortunately, those are the kinds of skills they will use for the rest of their lives, even if they decide not to start a career in a field that requires use of mathematics directly.

This lack of critical thinking in math classrooms is not always the fault of the teachers. They have standards that they must follow and skills that they are required to teach. While I believe that many teachers could spend more time making their lessons more student driven and more problem solving focused, sometimes it can be helpful to bring in an outside speaker to break the monotony of the every day classroom. To that end, the focus of this project is forming a relationship between the Colorado College math department and Colorado Springs schools. We developed problem solving activities, some more appropriate for 11th and 12th graders, some appropriate for all ages. In addition to introducing students to higher level math, these activities also fit in with common core standards, making it more likely that teachers will utilize them in their classroom. Our goal was to build critical thinking and problem solving skills in the students, but it was also to possibly unlock an interest in mathematics by showing them other interesting areas of math that they may be able to explore.

We begin by outlining a few simple activities and end by describing, in detail, a problem that goes against intuition and exposes students to a few advanced topics. The first activity involves a riddle about switching light bulbs on and off, with a solution that uses the fact that the square numbers have an odd number of factors. The second activity involves riddles about people guessing the color of a hat randomly placed on their head and introduces students to equivalence classes and the axiom of choice. The third activity introduces students to spherical geometry. The final activity relates two interesting problems: the proof of Goodstein's theorem and what happens when Hercules fights the Hydra dragon, which grows a certain number of heads when one is cut off. Both problems require use of second-order arithmetic.

2 Light Bulbs

If developing this kind of mathematical thinking can start early, it may keep more students interested in math and will allow them to think critically at later stages. There are plenty of activities that young students are capable of understanding, but they are not implemented because they seem too hard or because other things deemed more important take precedent. Working these activities in to math lessons will be critical for the development of our students. One activity that I did with fourth grade students revolved around the following problem:

One-hundred light bulbs with pull switches are lined up in a hallway, given a counting number based on their order. One-hundred men are lined up in front of light bulb number one, and each of the men are assigned a number based on their order in line as well. The first man walks through and turns every light bulb on. The second guy goes through and turns off every second light bulb. The third guy goes through a pulls the switch of every third light bulb, turning it off if it is on and turning it on if it is off. In general, the n^{th} guy pulls the switch of every n^{th} light bulb. After all 100 guys have walked through, what light bulbs will be on?

With this activity, the students can be introduced to several problem solving strategies, including minimizing the problem and drawing pictures. The pictures the students drew in our experience were similar to the table below.

Minimized to first 10 light bulbs

number assigned to guy walking through	1	2	3	4	5	6	7	8	9	10
1	on	on	on	on	on	on	on	on	on	on
2	on	off	on	off	on	off	on	off	on	off
3	on	off	off	off	on	on	on	off	off	off
4	on	off	off	on	on	on	on	on	off	off
5	on	off	off	on	off	on	on	on	off	on
6	on	off	off	on	off	off	on	on	off	on
7	on	off	off	on	off	off	off	on	off	on
8	on	off	off	on	off	off	off	off	off	on
9	on	off	off	on	off	off	off	off	on	on
10	on	off	off	on	off	off	off	off	on	off

After 10 guys have walked through
on:1,4,9
off: 2,3,5 6,7,8, 10

If they can make observation about the first 10 light bulbs after the first 10 guys have walked through, then make observations that they can generalize to the 100 person case, or the n person case. They will be able to see that 1, 4 and 9 will be on. From here, we can ask them to try to find a relationship between these three numbers. All of them will eventually be able to see that they are all square numbers. We take this opportunity to introduce the concept of a conjecture. We can now tell them to test their conjecture by extending their observations to the next square number, 16. When they see that it holds for the first four square numbers, now they can start thinking about why their conjecture might be true. The students are slowly being introduced to careful mathematical exploration and argument. It is not math focused on basic arithmetic and memorization, but rather on making observations about how numbers behave and how we solidify those observations.

At this point, it might be good to introduce the idea of what has to happen for a light bulb to be on and for a light bulb to be off. You can ask how many times must a switch be pulled for a light bulb to be on. It's on after being pulled once, three times, five times, etc. They can see it must be pulled an odd number of times. Now, we must ask what determines how many times the switch is pulled. This is when the large realization happens. It is is determined by the number of factors of the number. If a number has an odd number of factors, it's switch will be pulled an odd number of times. What numbers have an odd number of factors? Usually they are introduced to factors using factor pairs. A number $n = a \cdot b$. There is a certain group of numbers that have one pair such that a=b. That group of numbers is the squares. Students will usually start working with factors in 4th grade. Usually this means doing examples ad nauseam of finding all factor pairs of certain numbers. This activity allows them to work with factors while developing critical thinking. It also allows them to consider how factors behave.

3 Hats and the Axiom of Choice

The second activity was targeted towards 9th and 10th graders in basic algebra and geometry classes, though it can be appropriate for all ages. It was based on the following riddle:

A prison guard lines up 100 prisoners in a single-file line. Each prisoner is given either a red or a blue hat, but they do not know which color they received. Each prisoner can see all of the hats in front of them. The guard will start at the back and work his way forward, asking each prisoner to guess his hat color. Anyone who guess right will be set free, anyone who guesses wrong will be executed. The prisoners are given time before this game begins to strategize. What is the optimal strategy that will save the most prisoners?

At this point, many students went to a guessing strategy to start. Each prisoner has a 50 percent chance of receiving a red hat and a 50 percent chance of receiving a blue hat. With a guessing strategy and a large sample size, we would expect about half of the prisoners to survive. This is not the worst strategy, but it is also not the best. The next strategy that they would often fine would be for the first guesser to guess the color of the hat in front of them. It might also be their hat color, and it would ensure that the person in front of them would be saved. If every group of 2 does this, half of the prisoners will be saved for sure, while the other half will each have a 50 percent chance of surviving. This is an even better strategy, but still not quite the best.

When we got stuck as a group, I asked them to consider the same problem with just three people. In this case, we can consider all the possible arrangements of the hats. We have:

$$\{R, R, R\}$$
$$\{B, B, B\}$$
$$\{B, R, R\}$$
$$\{R, B, B\}$$
$$\{B, R, B\}$$
$$\{R, B, R\}$$
$$\{R, R, B\}$$
$$\{B, B, R\}$$

From here, we want the students to see that there are two cases that all of these arrangements can be split into: The case where the guy at the back of line is seeing two of the same colors in front of him and the case where he sees two opposite colors. He is not allowed to say "You two are wearing the same color hat." That is against the rules that the guard set. However, he knows that it is going to be one of the two cases, and he knows he is allowed to say two words. The students will eventually get that they can use the two guessing words as code words. For instance, assign "Red" to the case where the two front guys are wearing the same color hat and assign "Blue" to the case where the two front guys are wearing opposite colors. The outcome of this strategy is that each guy will be able to communicate what color the person in front of him is wearing.

Now that they have this idea of using code words, they are ready to tackle the 100 person case. The next step is to think about what the back guy is seeing in front of him. He sees 99 hats. Those 99 hats are going to be made up of m blue hats and n red hats. From basic arithmetic, we know that if $m + n = 99$, then if m is even, n is odd and if m is odd, n is even. Again, we have two cases. The students can use what they learned in the three person case to figure out how the back guy will tell all of his fellow prisoners the information he is seeing. They can agree to assign "Red" to the case where there are an odd number of red hats and "Blue" to the case where there are an odd number of blue hats. From here, all it takes is focus from the prisoners. If the back guy says "Red", for instance, then the next guy in line knows that if he also sees an odd number of red hats, then he must be wearing a blue hat. If he sees an even number of blue hats, then he must be wearing one of those odd red hats. Let's say he has a red hat. The next guy in line now knows that there must be an even number of red hats left. Again, if he sees an even number, he must be wearing blue, if he sees an odd number, he must have one of those even number of red hats. Each prisoner can keep track of whether there is an even or odd number of red hats left and use that information to correctly find his own hat color. This allows for 99 prisoners to be saved. This is the best strategy because the person at the back of the line has no way of gaining information about his hat color. He has a 50 percent chance that the code word he says is his actual hat color.

This is a great riddle that will challenge the critical thinking of students of all ages. However, there is more to be seen in variations of this riddle. For instance, the students might be surprised to learn that in the case of a countably infinite number of hats in which the prisoners are now wearing ear plugs, there is still a strategy that will ensure that only finitely many prisoners will guess wrong. First, recall the definition of a countably infinite set:

Definition A set is countably infinite if it is bijective to the set of natural numbers

The solution involves accepting the Axiom of Choice. The Axiom of Choice says that for every indexed family $\{X_{i,i \in I}\}$, there exists an indexed family of elements $\{x_i, i \in I\}$ such that $x_i \in X_i$ for every $i \in I$. Essentially, the Axiom of Choice says that there is a choice function that f such that $f(X_i) \in X_i$. This function allows us to choose exactly one element from each set in the family [1].

There is much that can be said about the axiom of choice. Its inclusion in this riddle provides an excellent opportunity to expose students to the idea and some of its consequences.

In their time of strategizing, the prisoners can agree to turn the possible arrangements of the hats into binary sequences. Assign 0 to blue and 1 to red. Then, they can define an equivalence relation that says that two of the sequences are equivalent if they are the same sequence after a finite number of entries.

Recall the definition of an equivalence relation:

Definition An equivalence relation is a relation that holds between two elements if and only if they are the member of the same cell of a partitioned set such that the intersection of all the cells is the empty set

In this case, the prisoners can create an equivalence relation that says two sets are equivalent if they are identical after a finite number of entries.

If $A =$ 0110001001101100101000101001100101001110**1010001110111010101010010101101...**

and $B =$ 1010100010101001010100010100100010010111**1010001110111010101010010101101...**

then $A \sim B$

This creates equivalence classes, which partitions the set of all possible hat arrangements. The prisoners can now invoke the Axiom of Choice to choose a representative sequence from each equivalence class. When the prisoners are lined up, they will see which equivalence class they are in. They will guess like they are in the representative sequence. After a finite number of guesses, every prisoner from then on will guess correctly. It turns out that this method will work even if we have a countably infinite number of possible colors, as they can just assign a unique natural number to each color [2].

This is likely their first introduction to equivalence classes, an idea which has important applications across mathematics.

4 Exploring Spherical Geometry

Students are introduced to plane geometry from a very early age, but unfortunately may never see any other geometries. Geometry is usually the first experience students have with writing proofs and thinking about math as an axiomatic system. One way to get them to explore this idea further is to introduce them to a non-euclidean geometric space. A great way to explore spherical geometry is to split the students into groups and give each of them a sphere to draw on. Make sure to define lines as great circles, then give them the following guide:

-Given a line and a point not on that line, try to construct a parallel line

-Construct a triangle, measure its angles

-What happens to the angle sum when you draw a really small equilateral triangle? What is the largest equilateral triangle you can construct?

-Construct a triangle with more than one right angle

-Try to construct two triangles of different sizes with the same angles

-What is the smallest number of sides that a polygon can have?

Right away, the issue of the parallel postulate is addressed. They will find that it is impossible to construct a parallel line in this space. They then see that a statement like the parallel postulate is not universally true, but dependent on the space you are working in. Next, they will think about the Euclidean Geometry theorem about the sums of the angles of triangles. The triangle angle sum theorem is often proven by constructing a parallel line through one of the vertices of the triangle and using the definition of supplementary angles and the alternate interior angle theorem. If not proven in this way, the proof involves some statement that is equivalent to the parallel postulate. Since we do not have parallel lines in spherical geometry, the students can guess that the triangle angle sum theorem will not hold. After measuring the angles of a constructed triangle, their observations should confirm their guess. They can then start to think about the largest and smallest possible angle sums. Constructing smaller triangles is similar to the idea of local linearity. They can understand that as the triangle gets smaller, the angle sum will approach 180 degrees. When the triangle gets larger, the curvature of the sphere takes over.

At this point, they are probably familiar with all of the triangle congruence acronyms. Now they can consider if those congruence hold in this new space, and which ones that did not hold in Euclidean will now hold. For instance, they should know that two triangles with three congruent angles are not necessarily congruent. But now they can see, with a little exploration, that it is impossible to construct two non-congruent triangles with three congruent angles in spherical geometry. So A-A-A is a congruence in this case.

The notion that there can be a polygon with less than three sides and that there can be a triangle with more than one right angle is ridiculous in the space they are familiar with. This activity allows them to see that the ideas they have learned in geometry are not inherently true, but dependent on the space and the assumptions made.

We end this activity with a brief description of hyperbolic geometry to compliment what they just explored.

5 Goodstein's Theorem

All of this led to a final activity involving sequences that go against intuition. Before we can describe the activity, we must become familiar with Goodstein Sequences and Goodstein's Theorem.

Definition The first term of a **Goodstein sequence** starts with a seed $k \in \mathbb{N}$ written in **hereditary base-2 form**. The nth term of a goodstein sequence is given by replacing all of the n in $G_{n-1}(k)$ with $n+1$, subtracting 1 from the entire term, and rewriting the term in **hereditary base-n+1 form**

Note that **hereditary base-n form** means that all exponents are rewritten in base-n form as well. For example, with a seed of 24, we would start with $24 = 3 * 2^3$ Then we would have $3 = 1 * 2^1 + 1 * 2^0$, which would give us $24 = 3 * 2^{2^1+1}$
Example Consider the Goodstein sequence with the seed 6:

$$G_1(6) = 2^2 + 2 \rightarrow \text{replace any 2 with a 3, subtract 1}$$
$$G_2(6) = 3^3 + 3 - 1 = 3^3 + 2 \rightarrow \text{replace any 3 with 4, subtract 1}$$
$$G_3(6) = 4^4 + 2 - 1 = 4^4 + 1 \rightarrow \text{replace any 4 with 5, subtract 1}$$
$$G_4(6) = 5^5 + 1 - 1 = 5^5 \rightarrow \text{replace any 5 with 6, subtract 1, rewrite in hereditary base-6 form}$$
$$G_5(6) = 6^6 - 1 = 5 * 6^5 + 7,775 = 5 * 6^5 + 5 * 6^4 + 5 * 6^3 + 5 * 6^2 + 5 * 6^1 + 5$$
$$\vdots$$

In this example, the exponents stop increasing after 5 terms. However, we can see that the base is going to grow for an unfathomably large amount of steps.

Theorem 5.1.1: Goodstein's Theorem Any goodstein sequence with a seed $k \in \mathbb{N}$ will terminate

This is certainly a surprising theorem. The goodstein sequence with seed 6 grew quickly, and this theorem says that no matter how big the seed is, the sequence will eventually terminate. Jeff Paris and Laurie Kirby proved that Goodstein's Theorem is not provable in peano arithmetic[4]. However, it is provable using second order arithmetic. In this paper, we will only prove it for certain cases. To do this, we will need some background information.

5.1 Set Theory Background

Definition Given an ordinal α, if $a < \alpha, a \in \alpha$

Definition β is a limit ordinal if and only if there is another ordinal less than β. Additionally, whenever there is an ordinal $\lambda < \beta$, there is another ordinal γ such that $\lambda < \gamma < \beta$

Definition Let ω be an ordinal such that if $k \in \mathbb{N}$, then $k \in \omega$. Then ω is a limit ordinal and the smallest infinite ordinal.

Definition A set A is well ordered if and only if every B $\subset$ A has a least element.

The set of natural numbers is well-ordered. Since ordinals are just an extension of the natural numbers, this leads us to the following proposition:

Proposition 5.1 The set of all ordinals is well-ordered

This is an important detail, because the following theorem will be an important factor in the proof of Goodstein's theorem:

Theorem 5.1.2 There can be no infinitely decreasing sequence within a well-ordered set

Proof Let A be a well ordered set with a least element c. Suppose there exists an infinitely decreasing sequence $B = b_1, b_2, ..., b_n, b_i \in A$. That is, $b_{k+1} < b_k$ for all $k \in \mathbb{N}$. Then there must be some $b_j = c$. Then, $b_{j+1} < b_j$ since B is an infinitely decreasing sequence. However, this contradicts c being the least element of A. Thus, B cannot be an infinitely decreasing sequence and there can be no infinitely decreasing sequence within a well-ordered set.

□

Corollary There is no infinitely decreasing sequence of ordinals

There is much that can be said about ordinal arithmetic. We will be using the definitions from P.R. Halmos's *Naive Set Theory* that we will need to understand how Goodstein Sequences will behave.

Ordinal Arithmetic

Addition: Given two ordinal numbers $\alpha = a_1, a_2, ...$ and $\beta = b_1, b_2, ...$, $\alpha + \beta = \{a_1, a_2, ..., b_1, b_2, ...\}$

We take every $b_j > a_i$. There is the possibility that the two sets are not disjoint, but we can solve this by showing that they are isomorphic to two disjoint sets. We can send $\alpha_i \to (0, \alpha_i)$ and $\beta_j \to (1, \beta_j$ Once we show that we can make them disjoint if needed, we will assume they are disjoint when doing arithmetic from now on.
Notice that $2 + \omega = \{0_2, 1_2, 0_\omega, 1_\omega, 2_\omega, ...\} = \omega$ but $\omega + 2 = \{0_\omega, 1_\omega, 2_\omega, ..., 0_2, 1_2\} \neq \omega$. The lack of commutativity will not cause any problems for our purposes, but it is important to notice.

Multiplication: Given the same α and β, $\alpha \cdot \beta$ is the same as adding α to itself β times. Again, we can make each individual alpha disjoint. Again, notice that $2 \cdot \omega = \{0_1, 1_1, 0_2, 1_2, 0_3, 1_3, ...\} = \omega$ but $\omega \cdot 2 = \{0_\omega, 1_\omega, 2_\omega, ..., 0_{\omega*}, 1_{\omega*}, 2_{\omega*}\}$ [1].

Exponentiation: This definition follows from the definitions of addition and multiplication.

Since Goodstein's Theorem says that all Goodstein sequences will eventually terminate, they must all eventually reach a peak and become decreasing sequences. That is hard to see by inspection. In order to prove this, we can construct a decreasing sequence, then match up each of its terms with the terms of any Goodstein sequence using a bijective relation. Constructing this a decreasing sequence is going to require use of infinite ordinals, which requires us to compare terms of a sequence of ifinite ordinals.

Lemma 5.1 $\omega^{n+1} > \omega^n * a_n + \omega^{n-1} * a_{n-1} + ... + \omega^1 * a_1 + \omega^0 * a_0$, where $0, 1, ..., n, n+1 \in \mathbb{N}$

Proof We will proceed by induction

$\omega^1 > \omega^0 \cdot a_0$ by definition of ω

Next, assume $\omega^n > \omega^{n-1} \cdot a_{n-1} + ... + \omega^1 \cdot a_1 + \omega^0 \cdot a_0$.

This implies that $\omega^n \cdot a_n + \omega^n > \omega^n * a_n + \omega^{n-1} * a_{n1} + ... + \omega^1 * a_1 + \omega^0 * a_0$

$\omega^{n+1} > \omega^n(a_n + 1) > \omega^n * a_n + \omega^{n-1} * a_{n1} + ... + \omega^1 * a_1 + \omega^0 * a_0$ [6]

5.2 Proof of Goodstein's Theorem

With this background information in place, we are only ready to prove Goodstein's Theorem for $G(k), k < 8$. We do not yet have the tools to deal with Goodstein Sequences that have continually changing exponents. The proof involves sending each term of a Goodstein sequence to a term in a sequence of ordinals by sending the base of the Goodstein sequence to ω. This would require us to layout ordering of infinite ordinals continually raised to infinite ordinals, which gets tricky. We have what we need to prove the $k < 8$ case, because every Goodstein sequence in this case reduces to the form:

$m^n * a_n + m^{n-1} * a_{n-1} + ... + m^1 * a_1 + m^0 * a_0$, $\{a_0, ..., a_n \in \mathbb{N}\}$ and $\{n \in \mathbb{N}\}$

To see this, consider $G(7)$:

$$G_1(7) = 2^2 + 2 + 1$$
$$G_2(7) = 3^3 + 3 - 1 = 3^3 + 2$$
$$.$$
$$.$$
$$.$$
$$G_4(7) = 5^5$$
$$G_5(7) = 6^6 - 1 = 6^5 * 5 + 6^4 * 5 + 6^3 * 5 + 6^2 * 5 + 6 + 5$$

This is manageable, because from this point on, the exponents of the sequence will not increase any more. We can handle this, because our lemma shows how infinite ordinals raised to natural numbers are ordered.
Before we jump into the proof, let us try to understand how Goodstein sequences could possibly terminate.

Consider $G(4)$:

$$G_1(4) = 2^2$$
$$G_2(4) = 3^3 - 1 = 2 * 3^2 + 2 * 3^1 + 2$$
$$G_3(4) = 2 * 4^2 + 2 * 4^1 + 1$$
$$G_4(4) = 2 * 5^2 + 2 * 5$$
$$G_5(4) = 2 * 6^2 + 2 * 6 - 1 = 2 * 6^2 + 6 + 5$$

We will advance 5 more terms before we exhaust the 2^0 piece; in the next term (the 11th term), we will have to rewrite the term in the hereditary base form from the definition of Goodstein sequences.

$$G_{10}(4) = 2 * 11^2 + 11$$
$$G_{11}(4) = 2 * 12 + 12 - 1 = 2 * 12^2 + 11$$

We will now advance 11 more terms before we exhaust the 2^0 piece, which will take us to the 22nd term. In the 23rd term, we will again have to rewrite the term so it fits the definition of a term of a Goodstein sequence (write it in hereditary base-24 form).

$$G_{22}(4) = 2 * 23^2$$
$$G_{23}(4) = 2 * 24^2 - 1 = 24^2 + 23 * 24 + 23$$

Continuing with this pattern, we can advance to the 46th and 47th terms, then the 94th and the 95th terms.

$$G_{46}(4) = 47^2 + 23 * 47$$
$$G_{47}(4) = 48^2 + 23 * 48 - 1 = 48^2 + 22 * 48 + 47$$
$$.$$
$$.$$
$$.$$
$$G_{94}(4) = 95^2 + 22 * 95$$
$$G_{95}(4) = 96^2 + 22 * 96 - 1 = 96^2 + 21 * 96 + 95$$

In general, when we hit a point in the sequence when we have $G_m(4) = (m+1)^2 + (m+1) * a_1 + 0 * a_0$, then $G_{m+1}(4) = (m+2)^2 + (m+2) * (a_1 - 1) + (m+1)$. We then advance to the term that is double the term we ended on, $G_{m+1}(4)$ in this case, which will take us to $G_{2m+2}(4)$. We then advance to the next term, subtract one, rewrite it according to the rules of Goodstein sequences, and advance to the term that is double the current term.

In our example, we left of at G_{95}. Let's consider how long it will take to completely exhaust the $a_1 * n^1$ piece. We will use this pattern 21 times. Meaning we will advance to the following term:

$(((((95 * 2 + 1) * 2 + 1) * 2 + 1) * ...) * 2 + 1) = 95 * 2^{21} + \sum_{k=0}^{20} 2^k = 201,326,589$. At this point, we will have

$$G_{201,326,589}(4) = 201,326,590^2 - 1 = 201,326,589 * 201,326,590 + 201,326,589$$

While the sequence has grown astronomically large, we notice that the degree has now decreased from 2 to 1. It is now of the form $G_m(4) = m * (m+1)^1 + m * (m+1)^0$. Using the same method we just used, we can reduce it to the form of $G_{m_1}(4) = (m_1 + 1)^1 + 0 * (m_1 + 1)^0$. Advancing, we will get $G_{m_1+1}(4) = m_1 + 2 - 1 = m_1 + 1$. To get $G_{m_1+2}(4)$, we follow the rules of Goodstein sequences and take replace $m_1 + 2$ with $m_1 + 3$. However, we no longer have a $m_1 + 2$, so nothing happens. We just subtract one as we move from term to term. This will happen over $m_1 + 1$ terms before the sequence terminates. To prove Goodstein's theorem for the case of $k < 8$, we will show that each Goodstein sequence with a seed $k < 8$ has a corresponding sequence of ordinals. This sequence of ordinals will be strictly decreasing, meaning they will eventually terminate to 0, which will imply that the corresponding Goodstein sequences will also terminate to 0.

Proof If n is the base of a term of a Goodstein sequence, define a mapping $f : n \to \omega$. We need to show that any two successive terms of a Goodstein sequence are sent to two terms, a_{n+1} and a_n, in the new sequence of ordinals such that $a_{n+1} < a_n$.

Let $G_{m-1}(k) = m^n * a_n + m^{n-1} * a_{n-1} + ... + m^1 * a_1 + m^0 * a_0$, $\{a_0, ..., a_n \in \mathbb{N}\}$

be the $m-1$ term of a goodstein sequence with a seed k. Then the next term is

$G_m(k) = (m+1)^n * a_n + (m+1)^{n-1} * a_{n-1} + ... + (m+1)^1 * a_1 + (m+1)^0 * a_0 - 1$

We now have to rewrite the m^{th} term in hereditary base-m+1 form. To do this, note that some of $a_0, ..., a_n$ could have a value of 0. Go to the smallest value of k such that $a_k \neq 0$ and omit all values such that $a_k = 0$. This gives us:

$G_m(k) = (m+1)^n * a_n + (m+1)^{n-1} * a_{n-1} + ... + (m+1)^k * a_k - 1.$

Writing this term in hereditary base-m+1 form,

$G_m(k) = (m+1)^n * a_n + (m+1)^{n-1} * a_{n-1} + ... + (m+1)^k * (a_k - 1) + (m+1)^{k-1} * m + (m+1)^{k-2} * m + ... + (m+1)^1 * m + m.$

Now, we must show $f(G_m(k)) < f(G_{m-1}(k))$

$f(G_{m-1}(k)) = \omega^n * a_n + \omega^{n-1} * a_{n-1} + ... + \omega^k * a_k$

$f(G_m(k)) = \omega^n * a_n + \omega^{n-1} * a_{n-1} + ... + \omega^k * (a_k - 1) + \omega^{k-1} * m + \omega^{k-2} * m + ... + \omega^1 * m + m$

Note that each term has $\beta = \omega^n * a_n + \omega^{n-1} * a_{n-1} + ...$, which gives us

$f(G_{m-1}(k)) = \beta + \omega^k * a_k$
$f(G_m(k)) = \beta + \omega^k * (a_k - 1) + \omega^{k-1} * m + \omega^{k-2} * m + ... + \omega^1 * m + m$

Notice that

$\omega^k * a_k = \omega^k * (a_k - 1) + \omega^k$, so $f(G_{m-1}(k)) = \beta + \omega^k(a_k - 1) + \omega^k$
.
Now, $\omega^k > \omega^{k-1} * m + \omega^{k-2} * m + ... + \omega^1 * m + m$ by **Lemma 5.1**, so $f(G_m(k)) < f(G_{m-1}(k))$.

Thus, we have sent an arbitrary pair of terms from an arbitrary Goodstein sequence of seed $k < 8$ to two terms in a decreasing sequence using a bijection. Since the new sequence will terminate, all Goodstein sequences must also terminate. The only way the sequence of ordinals can terminate is if the Goodstein sequence reaches a term with a base r and a value of $G_{r-1}(k) = r * 0 + b, b < r$. At this point, the Goodstein sequence will terminate in b more steps, as the change of base will no longer affect the value of the term. This must be the case since we know our sequence of ordinals must terminate by the corollary to **Theorem 5.1.2**.

6 Goodstein Sequences, Hercules, and the Hydra

High school students study sequences sometime during 9th and 10th grade when they are studying functions. This usually means dealing with simple arithmetic and geometric sequences. Our goal is to build off of that knowledge and introduce an interesting sequence that behaves in a surprising way, a way that goes against intuition. I introduced them to Goodstein sequences, starting with a simple seed of 3. The students notice that the sequence terminates rather quickly.

$$
\begin{aligned}
G_1(3) &= 2 + 1 \\
G_2(3) &= 3 + 1 - 1 = 3 \\
G_3(3) &= 4 - 1 = 3 \\
G_4(3) &= 3 - 1 = 2 \\
G_5(3) &= 2 - 1 = 1 \\
G_6(1) &= 1 - 1 = 0
\end{aligned}
$$

However, when the students inspect the next possible seed, 4, they notice that it will take this sequence much longer to terminate, if at all. $G_1(4) = 2^2$. The seed 3 did not have any 2^2 piece in it. Any seed with a 2^2 piece or higher powers of 2 will grow quickly, because the exponents will be growing with each new term.

$$
\begin{aligned}
G_1(4) &= 2^2 \\
G_2(4) &= 3^3 - 1 = 26 = 2 * 3^2 + 2 * 3^1 + 2 * 3^0 = 26 \\
G_3(4) &= 2 * 4^2 + 2 * 4^1 + 2 * 4^0 - 1 = 41 \\
&. \\
&. \\
&.
\end{aligned}
$$

The students are shocked when they see how big these sequences can get with larger seeds.

$$
\begin{aligned}
G_1(19) &= 2^4 + 2 + 1 = 2^{2^2} + 2 + 1 \\
G_2(19) &= 3^{3^3} + 3 + 1 - 1 = 7,625,597,484,990 \\
G_3(19) &= 4^{4^4} + 4 - 1 = 1.3 * 10^{154}
\end{aligned}
$$

When sequences become large quickly, it is hard to imagine that they ever might come back down. In fact, many students will use the evidence presented so far to conjecture that $G(k)$ will never terminate for any $k > 3$. Some students may question this conjecture, and may even be bold enough to make an opposite conjecture that all Goodstein sequences will terminate. After giving time to discuss the possibilities, you can introduce Goodstein's theorem to them. Then you can start discussing how you might prove it. Nothing in the proof is out of reach of a high school student:

1. The idea of ordinals and ω will be new to them. The definition of ordinals is straight forward, and they can think of ω as a way of "pinning down" the idea of infinity.

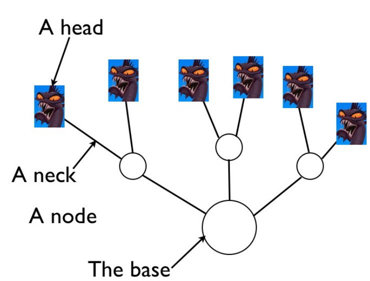
2. They will have worked with functions a lot up to this point, but the idea of a bijective relation may still be new to them. There are numerous simple examples you can provide for them, like matching each finger on one hand to another finger on the other hand.

3. Theorem 5.1.2 from the previous section will be new to them. The proof will be fairly easy to understand. You can even have them work together to construct a proof with your guidance. Understanding that there can be no infinitely decreasing sequence within a well-ordered set will be key.

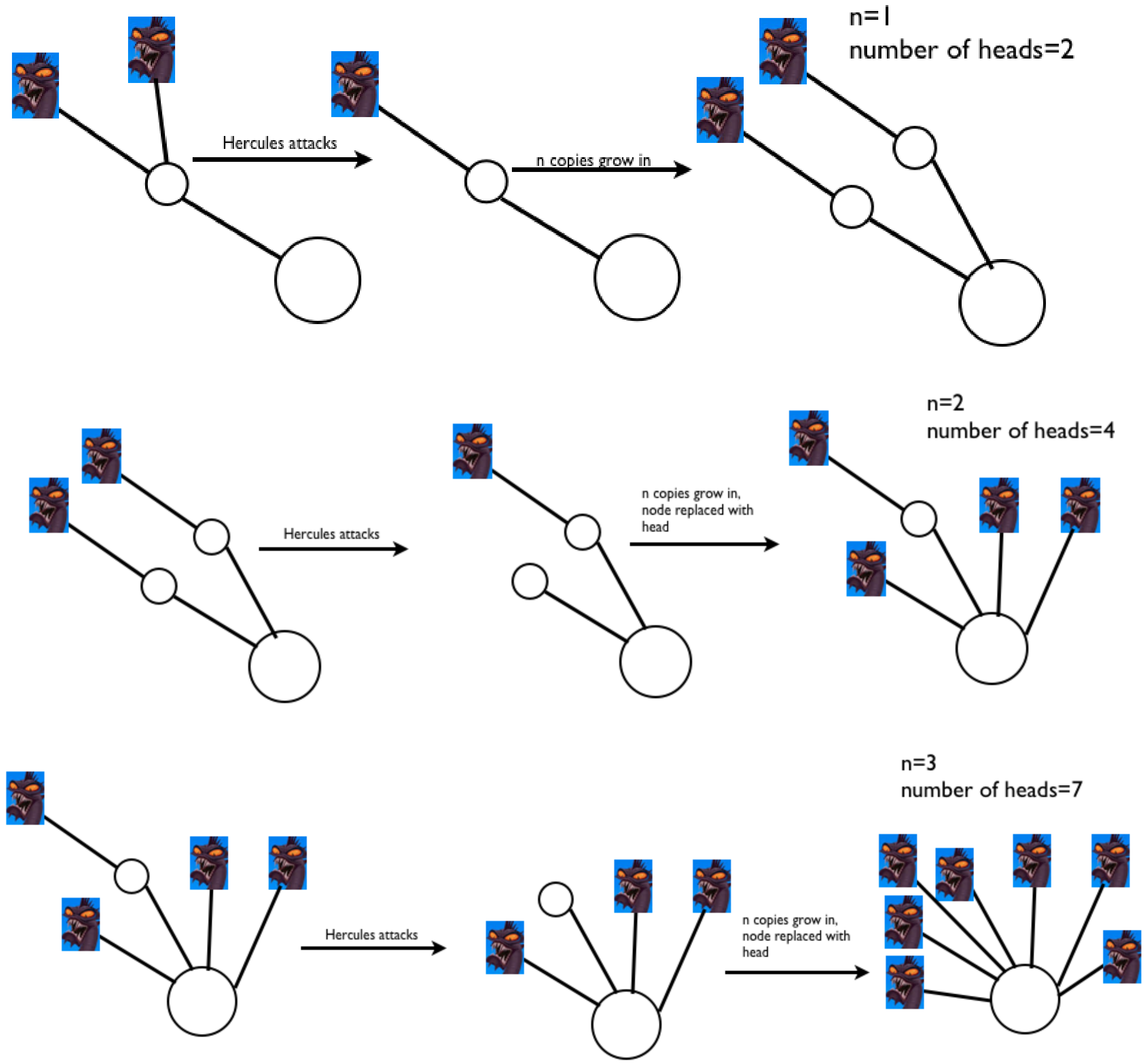
4. Getting into the non-commutativity of the ordinals is not important for the sake of this activity. You can just describe ordinal arithmetic as doing arithmetic with the order of the ordinals involved.

Some may take longer than others, but we believe every student has the ability to understand this proof. Once they have all of these pieces, you can work through the proof of Goodstein's theorem with them. The great thing about this activity is it has a connection to another interesting problem.

These results can be applied to an interesting question involving a story from Greek mythology: Hercules fighting the Hydra dragon. The story goes that any time Hercules would cut off a head of the dragon, more heads would grow back. In this problem, each Hydra will have a base, nodes connected to the base with a segment, and heads attached to nodes with necks/segments. When a head is cut off, go to the node it was attached to, move down to the next node(in some cases the next node is the base), and grow n copies of the branch the head was attached to from that node. Let n represent the number of times Hercules has attacked.

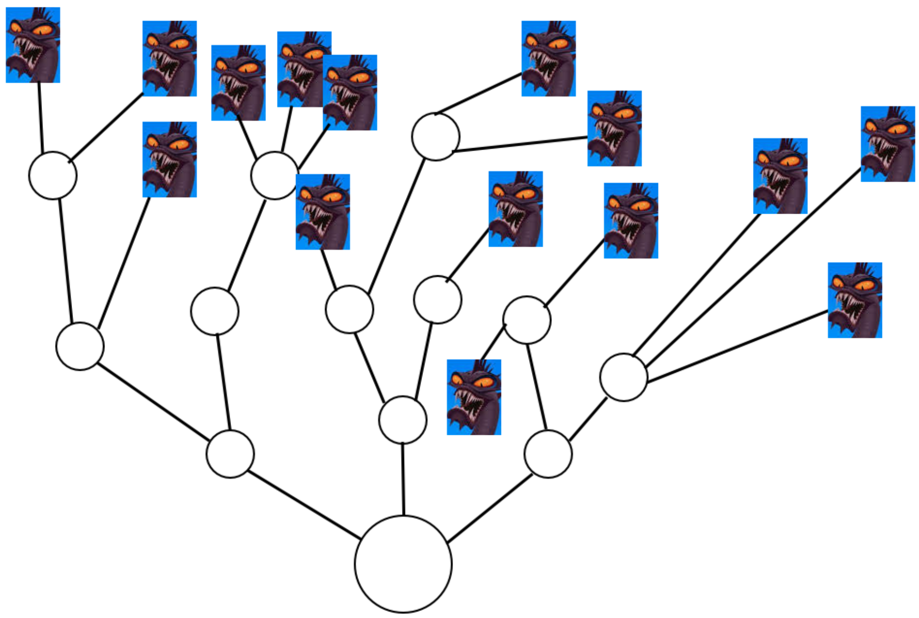


This is a very simple six-headed Hydra. We also notice that it is symmetrical. When the students consider if it is possible to defeat this Hydra, you can give them a chance to notice the symmetry. If they can see how one branch of the Hydra will be defeated, then they can be convinced that the whole thing can be defeated.



Eventually, each branch can be reduced to a certain number of copies of heads attached to the base by a single segment. The rules state that nothing grows back when a head is cut off from the base. Thus, this six headed hydra can be defeated.

Once everyone is comfortable with this, we introduce a more complex Hydra. Can this Hydra be defeated? What about a hydra with any number of heads, arranged in any possible way?

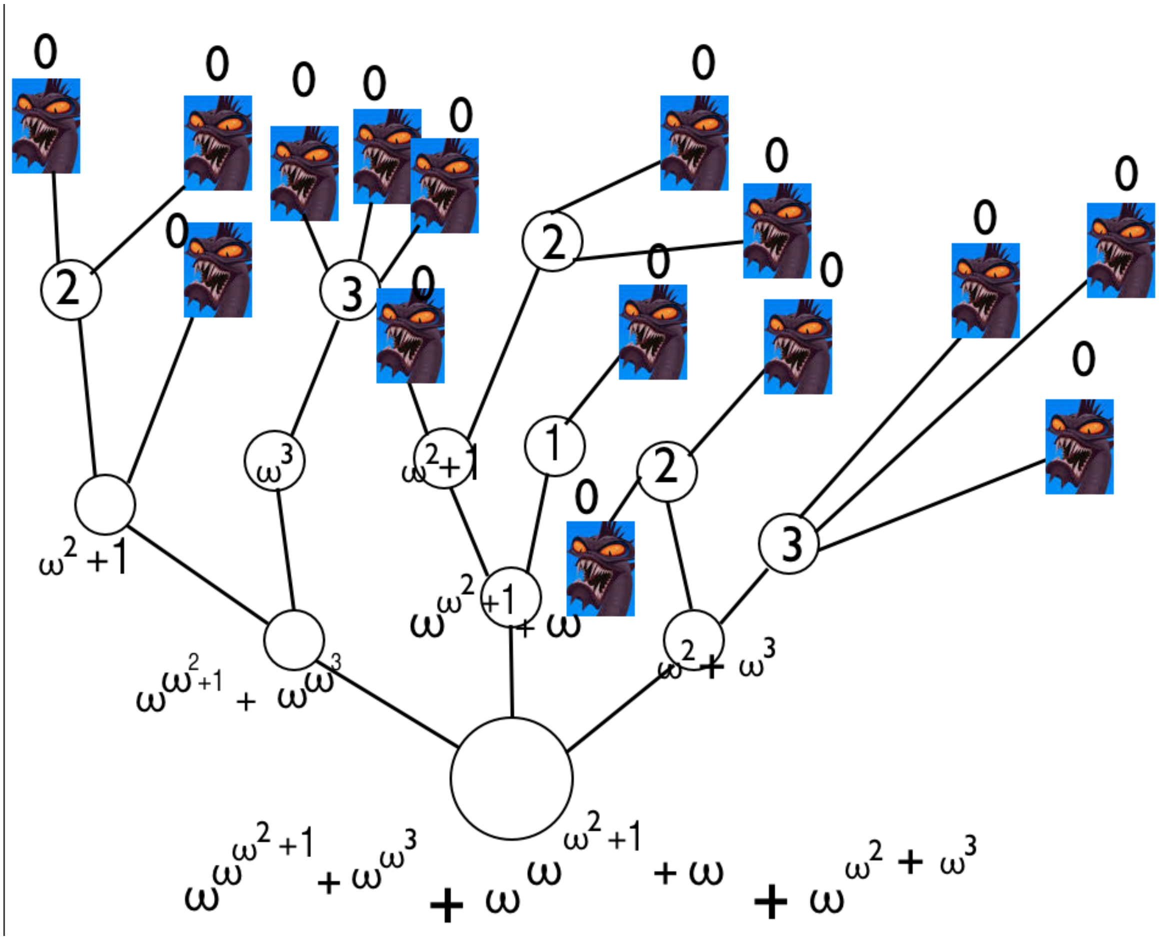


When the number of starting heads becomes large, the Hydra becomes too hard to deal with just by inspection. Fortunately, after working with goodstein sequences, we have a possible method for showing that all Hydras will eventually be defeated. It is not obvious, and the students might not get it right away. If we want to show that the number of heads goes to 0 as n gets large, one way to do that is to find a way to represent the Hydra as a decreasing sequence of ordinals. If the sequence goes to 0, then the number of heads will go to 0.

We will define the sequence using the following rules:

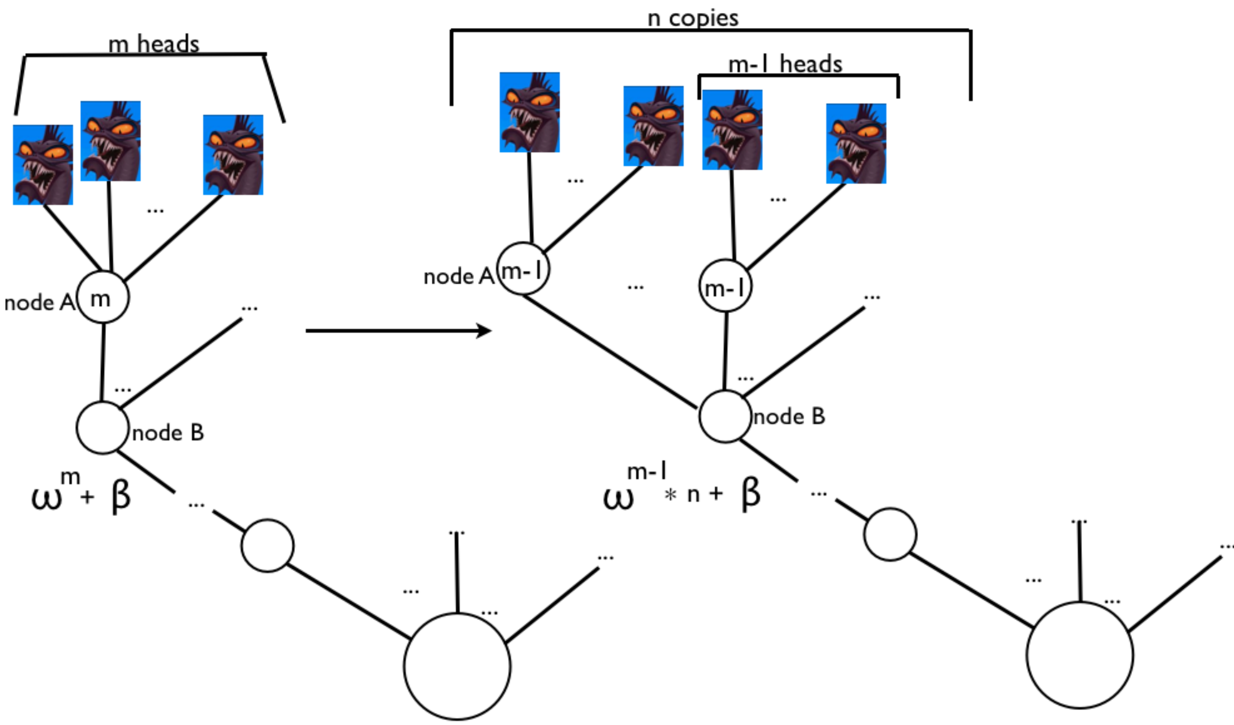
1. Each head will be labeled with a "0"
2. Each node will receive a value based on what is attached to it from above. If it has a node with a value of β attached to it from above, it receives a value of ω^{β}.
3. If it has multiple segments attached to it, those segments will be attached with a plus sign. If a node has segments leading to values of α_1,α_2, ..., α_n, then the node receives a value of $\omega^{\alpha_1}+...+\omega^{\alpha_n}$[5]

Using these rules, we can label our more complex hydra:



The question is, what happens when a head is cut off? If a node A has m heads attached to it, then the node A will have a value of m. The node B below it will have a value of $\omega^m + \beta$, where β depends on the other segments attached to B. When a head is cut off from A, it will now have a value of $m - 1$. Then, n copies of that branch will grow from node B. So now, node B will have a value of $\omega^{m-1} * n + \beta$, which we showed in the last section is less than $\omega^m + \beta$.

Therefore, every node below node A will have a smaller term after the head is cut off, and the overall term of the sequence connected to the base must then be smaller. This gives us a decreasing sequence of ordinals. This idea is similar to the proof of goodstein's theorem. We have a one-to-one relation between the different stages of the Hydra and the decreasing sequence of ordinals. Therefore, even though cutting off a head will lead to more heads in the next turn until Hydra reaches its peak, we know that the number of heads will eventually be 0 since the sequence of ordinals representing the Hydra terminates.



7 Conclusion

The Common Core State Standard state that by the end of fourth grade, students must be able to "Find all factor pairs for a whole number in the range 1 to 100, recognize that a whole number is a multiple of each of its factors, and determine whether a given whole number in the range 1100 is a multiple of a given one-digit number."

Just finding all the factor pairs of given numbers will develop this skill, but it can get boring for the students. The light bulb activity allows them to work with factors, think deeply about how they work, and develop critical thinking skills.

The standards state that high school students will focus on Euclidean Geometry. "Although there are many types of geometry, school mathematics is devoted primarily to plane Euclidean geometry, studied both synthetically (without coordinates) and analytically (with coordinates). Euclidean geometry is characterized most importantly by the Parallel Postulate, that through a point not on a given line there is exactly one parallel line. (Spherical geometry, in contrast, has no parallel lines.)" However, most students do not understand the statement "Euclidean geometry is characterized most importantly by the Parallel Postulate," because to them, parallel lines are something that exist. They take them for granted, as well as every geometric theorem that is equivalent to the parallel postulate. Introducing them to spherical geometry will allow them to think about how axiomatic systems work. It will allow them to think about all of the theorems they have been using blindly in a different way.

The standards state that students who have studied functions must be able to "Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers." This is another topic that can become stale. The Goodstein activity introduces an interesting sequence which behaves in a way that goes against intuition. The proof of Goodstein's theorem can be applied to the Hercules problem, which will be engaging for them and may spark an interest in studying sequences. Additionally, it introduces them to an interesting application of set theory, which they normally would not see in high school.

Right now math is seen as a subject that is only accessible to people born with natural math abilities. For others, it can be a dreadful and boring subject. Part of that is due to the way we teach it. It is not fair that only some students have access to some of the real beauty of the subject while others have to go through life never experiencing the pieces of math that have kept humans interested in the subject for thousands of years. If we expect more out of all of our students, not just the gifted ones, we can spark an interest in math in more students throughout the country. Further, the students who choose not to study math beyond secondary school will be left with a more developed brain with critical thinking and problem solving abilities. A focus on developing these skills will always give teachers a good answer to the question "But when we will ever use this?", because it will be used anytime critical thinking is required. Working through different areas of math prepare your brain to take on difficult problems in all areas of the world. If that is not important, then we better stop wasting money sending our children to school.

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