The Cantor Set as a Fractal and its Artistic Applications

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Monday 13th April, 2015

Abstract

The Cantor middle-thirds set is an interesting set that possesses various, sometimes surprising mathematical properties. It can be presented through ternary representation and obtained through an iterative process. This paper will discuss selected topological properties of the Cantor set, as well as its connection to fractal geometry. It will then discuss the existence of the Cantor set in a variety of artistic contexts.

1 Introduction

Georg Cantor (1845-1918) was a German mathematician and the creator of transfinite set theory (Dauben 1). Cantor's work was often regarded as controversial, partially because of the use of infinity in his mathematics (Dauben 1). He was also the first to publish the traditional middle-thirds set, which we refer to as the Cantor set. Though the Cantor set was an abstract concept at the time of its publication in 1883, Cantor explored many of its deep mathematical qualities. The Cantor set is a fractal and can be achieved through use of dynamical systems. The problem of the dynamics of iteration and fractals was briefly explored in the early 19th century, but it was not until the use of computers that it was developed in more depth (Mandelbrot 23). Here, we will discuss some of the topological properties of the Cantor set. We will consider the Cantor set as both a one-dimensional and two-dimensional dynamical system. Lastly, we will discuss the Cantor set as a fractal.

Benoit Mandelbrot developed fractal geometry in the 1970's. He referred to his math as a new "geometric language" (Mandelbrot 21). People were slow to accept the new mathematical concept of fractals, but eventually Mandelbrot published a paper about his findings (Mandelbrot 22). Mandelbrot considered fractals to be artistic objects. Here, we will discuss the connection between the Cantor set fractal and art. We can find resemblance to fractals, particularly the Cantor set, in many artistic contexts. We will focus on its presence in architecture and Chinese art. These connections to art make a fascinating topic in mathematics applicable in a non-scientific context.

1.1 The Cantor Middle-Thirds Set

The traditional Cantor middle-thirds set is constructed through an iterative process. Beginning with the closed set $[0,1]$, the open middle third $\left(\frac{1}{3},\frac{2}{3}\right)$ is removed. Two closed sets remain. The middle third is then removed from each of these sets, namely the intervals $\left(\frac{1}{9},\frac{2}{9}\right)$ and $\left(\frac{7}{9},\frac{8}{9}\right)$. This process is repeated infinitely many times, and the set that remains is the Cantor middle-thirds set. More formally, consider the sets $I_0, I_1, I_2...$, where

$$I_0 = I = [0,1]$$

$$I_1 = I \backslash \left(\frac{1}{3},\frac{2}{3}\right)$$

$$I_2 = I_1 \backslash \left(\frac{1}{9},\frac{2}{9}\right) \bigcup \left(\frac{7}{9},\frac{8}{9}\right) ...$$

We define the Cantor set to be $C = \bigcap_{k=0}^{\infty} I_k$, or the intersection of $I_0, I_1, I_2...$ We can illustrate C by depicting each iteration of removing middle-thirds on a separate line (Figure 1).

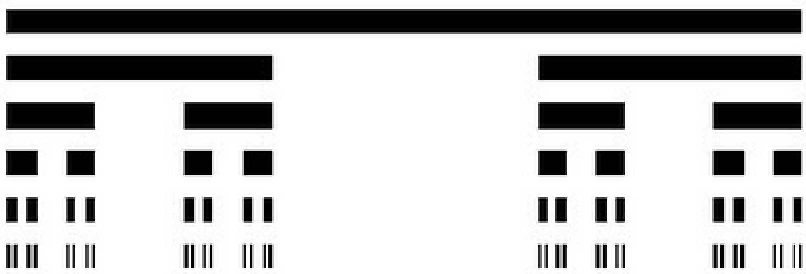

Figure 1: Typical representation of the Cantor Set, Tex Stack Exchange.

After the first iteration, the Cantor set consists of two disjoint intervals of length $1/3$. After the second iteration, the Cantor set consists of 4 disjoint intervals of length $1/9$. At the kth iteration, the Cantor set consists of 2^k intervals of length $1/3^k$.

Proof. Proceeding by induction, we consider $I_0 = [0, 1]$:

$$\frac{1}{3^0} = \frac{1}{1} = 1.$$

At this iteration, C has $2^0 = 1$ interval. Now, assume that the set I_k has 2^k disjoint intervals of length $1/3^k$. If we remove the middle third from an interval, each subinterval will be one-third the length of the original interval:

$$\frac{1}{3^k} \cdot \frac{1}{3} = \frac{1}{3^{k+1}}.$$

Also, the 2^k intervals are all split into two intervals:

$$2^k \cdot 2 = 2^{k+1}.$$

By induction, I_k consists of 2^k disjoint intervals of length $1/3^k$. □

We have described the classic middle-thirds Cantor set. However, note that any set that is constructed by an iterative process of removal of some constant portion of the set can be considered a Cantor set.

2 Topological Properties

2.1 Ternary Representation

The Cantor middle-thirds set can be expressed through ternary representations. Recall that a geometric series $\sum_{i=0}^{\infty} a^i = 1 + a + a^2 + a^3...$ converges absolutely to $\frac{1}{1-a}$ if $|a| < 1$. Consider the series $\sum_{i=1}^{\infty} \frac{s_i}{3^i}$. Suppose that each s_i is either $0, 1$ or 2. Then the series $\sum_{i=1}^{\infty} \frac{s_i}{3^i}$ is dominated by the convergent geometric series $\sum_{i=1}^{\infty} \frac{2}{3^i} = 1$. Thus, by the Comparison Test, $\sum_{i=1}^{\infty} \frac{s_i}{3^i}$ converges and $0 \leq \sum_{i=1}^{\infty} \frac{s_i}{3^i} \leq 1$.

Ternary Expansion We call $0.s_1 s_2 s_3...$ the *ternary expansion* of x if $x = \sum_{i=1}^{\infty} \frac{s_i}{3^i}$, where each s_i is either $0, 1$ or 2.

We claim that every $x \in [0, 1]$ has a ternary expansion. Let s_1 be the largest among $0, 1, 2$ for which $x \geq \frac{s_1}{3}$. Then, pick the largest s_2 for which $x - \frac{s_1}{3} \geq \frac{s_2}{3^2}$. Proceed inductively to get the largest s_n for which $x - \sum_{i=1}^{n-1} \frac{s_i}{3^i} \geq \frac{s_n}{3^n}$. Then note that $x - \sum_{i=1}^{n-1} \frac{s_i}{3^i} \leq \frac{1}{3^n}$ and so, we see that the infinite series $\sum_{i=1}^{\infty} \frac{s_i}{3^i}$

will converge to x.

We claim that each point x of the Cantor set can be represented as a ternary expansion $0.s_1s_2s_3...$ where each s_i is 0 or 2. If x has a ternary expansion for which some $s_i = 1$, then x lies in a middle third interval that has been removed. This is because x would be past the left third interval, but it would not yet reach the right third interval. For example, if $s_1 = 1$, $x \neq 1/3$ will be greater than 1/3=.1, but it will not yet reach 2/3=.2, placing it in a middle third. This idea can be applied to any s_i. Thus, no Cantor set element ternary expansion contains a 1, excluding the endpoints, which may have a 1 as the right-most digit of their ternary expansion. In this case, x has an alternative expansion that contains no 1's. For example, the ternary representation for 1/3 is .1 and is equivalent to the representation .0222.... So, we can consider the Cantor set to be the set of real numbers in the unit interval $[0, 1]$ with ternary representations containing only 0's and 2's.

Similarly, we can represent any x in $[0, 1]$ by a binary expansion $\displaystyle\sum_{i=1}^{\infty} \frac{s_i}{2^i}$ consisting of 0's and 1's. We will use this expansion in the next section.

2.2 Uncountable

When we consider the construction of the Cantor set, it seems like we "throw out" most points of the unit interval. Intuitively, we would think that C should be a small set. The fact that the Cantor set is actually uncountable is one of the surprising topological properties of the set. We will prove this here:

Proof. If x is in the Cantor set, it has a unique ternary expansion using only 0's and 2's. By changing every 2 in the expansion of x to a 1, the ternary expansions of the Cantor set can be mapped to binary expansions, which have a one-to-one correspondence with the unit interval. This can also be done in the opposite direction to map binary expansions to ternary expansions. The only exceptions to this correspondence are the binary expansions ending in infinitely many 0's or 1's and the ternary expansions ending in infinitely many 0's or 2's. However, these exceptions are countable because there are finitely many ways to begin a binary representation before ending in an infinite string of 0's or 1's, and there are finitely many ways to begin a ternary representation before ending in an infinite string of 0's or 2's. Thus, there is a one-to-one correspondence between the binary and ternary exceptions. Since each real number in $[0, 1]$ can be represented as a binary expansion, the Cantor set has a one-to-one correspondence with the unit interval. Now, $[0, 1]$ is uncountable, and so the Cantor set is uncountable. □

2.3 Closed, Perfect, and Compact

Here, we will discuss why the Cantor set is closed, perfect, and compact. By construction, each I_k is closed because it is the complement of an open set. Thus, $\displaystyle\bigcap_{k=0}^{\infty} I_k$ is closed because the intersection of closed sets is also closed. Therefore, the Cantor set is a closed set. We will now see that the Cantor set is perfect.

Isolated Point Point x in set S is an *isolated point* if ϵ-ball $B(x, \epsilon)$ surrounding x does not contain another point in S.

Perfect Set S is *perfect* if it contains no isolated points.

We claim that the Cantor set is perfect.

Proof. Consider $x \in C$. For any ϵ, we have the open ball $B(x, \epsilon)$. We can choose k so that $\dfrac{1}{3^k} < \epsilon$. Let I_k be the union of 2^k disjoint intervals of length $1/3^k$. Then, $x \in I_k$. Let x be in subinterval $s \in I_k$, and then $s \subseteq B(x, \epsilon)$. In the $k + 1$ iteration, s is split into subintervals a and b. Let x be in a. By self-similarity, we know that there must be points of C in b. Thus, there are points of C in $B(x, \epsilon)$ not equal to x, and x is not an isolated point. Therefore, no point in C is an isolated point, and C is perfect. □

Now, recall that the unit interval $[0, 1]$ is closed and bounded. Thus, it is compact by the Heine-Borel Theorem (Ross 90). We see that the Cantor set is compact because every closed subset of a compact space is compact (Willard 119). We have now shown that the Cantor set is closed, perfect, and compact.

2.4 Totally Disconnected

We also can prove that the Cantor set is totally disconnected.

Totally Disconnected A set is *totally disconnected* if it contains no subintervals.

This is another non-intuitive property of the Cantor set. We have already proved that C is perfect, or has no isolated points. We would then expect the Cantor set to contain subintervals. Here, we will prove this to be false.

Proof. Consider $a, b \in C$. Recall that I_k consists of finitely many disjoint intervals of length $1/3^k$. We can find k where $\frac{1}{3^k} < |b-a|$. So, if the distance between a and b is more than $1/3^k$, a and b must belong to different subintervals of I_k. By the construction of I_k, there must be an interval in (a, b) that is not in I_k. Thus, there exists $z \notin I_k$ with $a < z < b$. Therefore, I_k does not contain (a, b). Since $C = \bigcap_{k=0}^{\infty} I_n$, C does not contain any interval (a, b). Thus, C is totally disconnected. $\square$

Each of the previously discussed topological properties relate to an important theorem (Willard 217): the Cantor set is the only totally disconnected, perfect, compact metric space (up to homeomorphism). This is an interesting theorem that requires more complicated topology than we have discussed, so we will not prove it here.

3 Cantor Set as a Dynamical System

We have discussed the traditional construction of the Cantor set and some of its topological properties. We can also reach the Cantor set through the use of dynamical systems. We will explore two different ways this can be achieved.

3.1 Iterated Function System

The Cantor set can be produced by the iteration of a function system.

Consider the two linear functions (Devaney, 192) from $\mathbb{R}^2 \to \mathbb{R}^2$:

$$A_0 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$A_1 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} x-1 \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

.

We claim that if x is an element of the Cantor set, then this iterated function system will send the point (x, y) to another point of the form (c, y_1) where c is in the Cantor set. In other words, this system fixes the Cantor set. Recall that the Cantor set consists of all points in the interval $[0, 1]$ with ternary expansions containing only 0's and 2's. We see that A_0 shrinks the x-coordinate by $1/3$, and its corresponding ternary representation by .1. For example, consider $x = .0022022 \in C$. A_0 would shrink x by .1 to .00022022. This ternary representation consists of only 0's and 2's, so it is still contained in the Cantor set. A_1 shrinks the x-coordinate by $1/3$ and shifts it by $2/3$, or by its ternary representation of .2. For example, A_1 would shift $x = .0022022$ to .20022022. This ternary representation also consists of only 0's and 2's, so it is still contained in the Cantor set. So, we see that the system of equations A_0, A_1 takes points of the Cantor set back into the Cantor set. These functions, no matter the order they are performed, leave the Cantor set fixed.

The Cantor set is called the *attractor* of this iterated function system. This means that any point in the plane with any y-coordinate will eventually be "pulled into" the Cantor set when this function system is applied. That is, after enough iterations, every point in the plane will converge to a point $(c, 0)$ where c is in the Cantor set. To see this, we consider any number of iterations of A_0 and A_1 in a random order. We can represent this random sequence of choice of A_0 or A_1 by a sequence $(s_1 s_2 s_3 ... s_n)$ where each s_i

is either 0 or 2 representing the application of A_0 and A_1 respectively. Now let $x \in \mathbb{R}$, and let x_n be the result of the applied sequence of A_0 and A_1. We see that:

$$x_n = x \cdot \frac{1}{3^n} + \frac{s_1}{3^{k-1}} + \frac{s_2}{3^{k-2}} + ... + \frac{s_n}{3^{k-n}}.$$

When we take the limit of x_n as $n \to \infty$, we see that the first term $x/3^k$ approaches 0. The remainder of this expression is of the form $\sum_{i=1}^{\infty} \frac{s_i}{3^i}$ with each s_i equal to 0 or 2, which we know means it is an element of the Cantor set. The y-coordinate will be sent to 0. Thus, we see that any point in the plane will converge to a point $(c, 0)$, where c is in the Cantor set, after enough iterations of A_0 and A_1 in a random order.

3.2 Iterated Tent Function

We can also produce the Cantor set by a different dynamical system. To illustrate this, we consider the Tent Function:

$$T(x) = \begin{cases} 3x & \text{if } x \leq 1/2 \\ 3 - 3x & \text{if } x > 1/2 \end{cases}$$

We claim that by iterating this function, the points that are not sent to infinity are exactly the Cantor set.

If $x < 0$, then $T(x) < 0$. At the next iteration of the Tent Function, $T^2(x) = 9x < T(x)$. At the third iteration, $T^3(x) = 27x < T^2(x)$. We see that as $n \to \infty$, $T^{(n)} \to -\infty$ for $x < 0$.

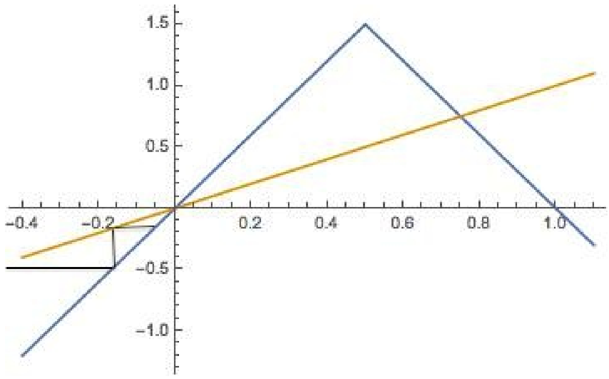

Figure 2: Tent Function, map of point $x < 0$

Graphically, we trace a point from the tent map to the line $y = x$. We begin with a graph of the tent function and the function $y = x$. At an x value outside of our interval $[0, 1]$, we map from $y = x$ to the tent function. At that y value, we map back to $y = x$. Then, from that x value, we map back to the tent function. By repeating this process, we see that the point we have been mapping goes to negative infinity. Figure 2 depicts this process for $x < 0$.

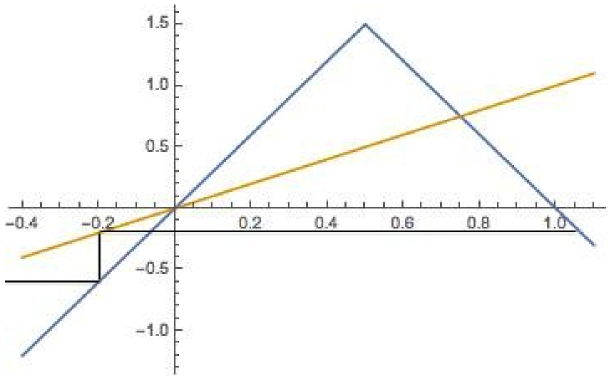

Figure 3: Tent Function, map of point $x > 1$

If we choose $x > 1$, after the first iteration $T(x) < 0$. Recall that as $n \to \infty$, $T^{(n)}(x) \to -\infty$ for $x < 0$. Therefore, for $x > 1$, x is sent to $-\infty$, as depicted in Figure 3. Thus, we find that for $x \notin [0, 1]$, x is sent to negative infinity by iterating the Tent Function.

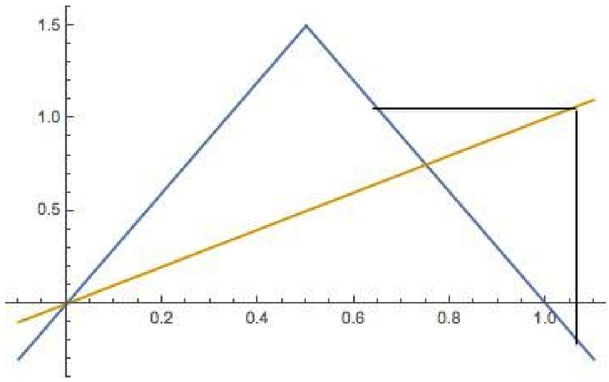

Figure 4: Tent Function, map of point $\frac{1}{3} < x < \frac{2}{3}$

By this same process, we also see that if $x \in \left(\frac{1}{3}, \frac{2}{3}\right)$, then x is sent to infinity (Figure 4). For example, $T(1/2)$=3/2. We saw above that this is eventually sent to $-\infty$ because $x > 1$.

In fact, any point in a middle third will be sent to $-\infty$. For example, if $x \in \left(\frac{1}{9}, \frac{2}{9}\right)$, then it is sent to $\left(\frac{1}{3}, \frac{2}{3}\right)$ in one iteration of the Tent Function, which we discussed above. This is supported algebraically: if $1/9 < 2/9$ then $1/3 < T(x) = 3x <$2/3. This is true for any x in a middle third interval.

Any point that is in the Cantor set, with the exception of the endpoints, will be sent back to itself after enough iterations of the Tent Function. We will not provide a formal argument for this, but we will explore an example. Consider 3/13, which is not an endpoint. Now, we will see that 3/13 is sent back to itself after three iterations of the Tent Function:

$$T\left(\frac{3}{13}\right) = 3\left(\frac{3}{13}\right) = \frac{9}{13}$$

$$T^2\left(\frac{3}{13}\right) = 3 - 3\left(\frac{9}{13}\right) = \frac{12}{13}$$

$$T^3\left(\frac{3}{13}\right) = 3 - 3\left(\frac{12}{13}\right) = \frac{3}{13}$$

The ternary representation of 3/13 is .02002..., which confirms that it is in the Cantor set.

We also see that at the x values 0 and 3/4, there is no line to be mapped graphically. These points are where the tent function and $x = y$ intersect, and are called fixed points. This is supported algebraically:

$$3(0) = 0$$

$$3 - 3\left(\frac{3}{4}\right) = \frac{3}{4}.$$

If we consider the endpoints of the Cantor set intervals, we find that they eventually are attracted to the fixed point 0. These are called *eventual fixed points*. For example, endpoint 1/3 is attracted to the fixed point 0 after the second iteration of the Tent Function:

$$T\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right) = 1$$

$$T^2\left(\frac{1}{3}\right) = 3 - 3(1) = 0$$

We will now prove that each endpoint is an eventual fixed point and is sent to 0:

Proof. All endpoints of the Cantor set are of the form $\sum_{k}^{n} \frac{s_k}{3^k}$ because they must be rational. Recall that the endpoints can contain a 1 in the right-most digit place, but these can be rewritten in terms of 2's. That is, $s_n = 1$ is possible if x is an endpoint. If $s_1 = 0$ then $T(x) = 3x$. This shifts the left-most ternary digit left by 1. If $s_1 = 2$, then $T(x) = 3 - 3x = 3 - 2.s_2s_3...s_n = 3 - (2 + .s_2s_3...s_n) = 1 - .s_2s_3...s_n$. We see that $T(1 - .s_2s_3...s_n) = T(.s_2s_3...s_n)$ because $T(x)$ is symmetrical about the line $x = 1/2$. We can repeat this process until we reach s_n. If $s_n = 0$ or 2, then we repeat one last time, and we reach 0. If $s_n = 1$, we apply $T(x)$ again and reach 1, which is sent to 0 by another iteration of $T(x)$. Thus, We see that the endpoints are eventually sent to the fixed point 0. ☐

Here, we see an example of this process. Consider the endpoint 7/9=.21.

$$3(.21) = 2.1$$

$$T(.21) = 3 - 2.1 = 3 - (2 + .1) = 1 - .1$$

$$T(1 - .1) = T(.1) = 1$$

$$T(1) = 3 - 3(1) = 0$$

We see that the endpoints are eventually sent to the fixed point 0. The endpoints are not sent to infinity, which means they are part of the Cantor set. This correlates with our analysis of their ternary representations in Section 2.1.

Iterating the Tent Function sends all points of C back to themselves or to a fixed point. Thus, we see that iterating the Tent Function fixes exactly the Cantor set.

3.3 The Cantor Set is a Fractal

The classic Cantor middle-thirds set is a mathematical object called a fractal.

Fractal A *fractal* is a subset of $\mathbb{R}^n$ that exhibits self-similarity on all scales and has fractal dimension. A fractal does not necessarily have topological dimension.

Informally, self-similarity means that we can apply a rescaling function to the set and the image of the set will look the same. Benoit Mandelbrot provided an informal definition of a fractal: "Fractals are geometric shapes that are equally complex in their details as in their overall form. That is, if a piece of a fractal is suitably magnified to become of the same size as the whole, it should look like the whole, either exactly, or perhaps only after a slight limited deformation" (Mandelbrot 22). We can see that the Cantor set is self-similar by examining C at a different scale. Recall that I_1 consists of two intervals of length 1/3. If we magnify one of these subintervals by 3 and continue the process of removing the middle third, we see that we have an exact copy of the full-scale Cantor set.

3.4 Fractal Dimension

We will now discuss the difference between fractal and topological dimension.The type of dimension that we are most familiar with is topological dimension. A point is of dimension 0, a line is one-dimensional, a square is two-dimensional, and a cube is three-dimensional. Logically, we understand these dimensions as the number of "linearly independent" directions we can move along an object (Devaney 185). For example, we can move along the length and width of a square, so we understand it to be two-dimensional. We define topological dimension here:

Topological Dimension k An open set S has *topological dimension* k if each point in S has an arbitrarily small neighborhood homeomorphic to $\mathbb{R}^k$ (Devaney 186).

For example, an open square has topological dimension 2 because the points in a square have arbitrarily small neighborhoods that are two-dimensional.

Notice this applies when $k = 0$. In that case, every point in the set has a neighborhood that is homeomorphic to a zero-dimensional object, such as a point. For example, a discrete set has dimension 0.

It remains to show that the Cantor set has fractal dimension. Finding the dimension of the Cantor set is more complicated then finding the dimension of simpler objects. We proved that C contains no subintervals. This implies that the Cantor set contains no point with a neighborhood that is homeomorphic to $\mathbb{R}^1$. Thus, the Cantor set is not one-dimensional. However, C is also perfect and contains no isolated points, so it does not have dimension 0. Therefore, the Cantor set has dimension in between 0 and 1. We can think of the Cantor set as somewhere in the middle of unconnected isolated points and pieces of straight lines (Peak, Frame 92). At every scale, C appears to be linear stretches, though we know that each of these stretches is broken up at the next iteration (Peak, Frame 92). To consider the dimension of the Cantor set, we must define a new type of dimension: *fractal dimension.* First, we must note that only sets that are affinely self-similar have a well-defined fractal dimension (Devaney 186).

Affine Self-similar A set S is called *affine self-similar* if S can be subdivided into k congruent subsets, each of which may be magnified by a constant factor M to yield a whole set S (Devaney 187).

As we discussed in Section 3.3, the Cantor set is affine self-similar.

Fractal Dimension Suppose the affine self-similar set S may be subdivided into k congruent pieces, each of which may be magnified by a factor of M to yield the whole set S. Then, the *fractal dimension* D of S is (Devaney 188):

$$D = \frac{\log k}{\log M}.$$

To understand fractal dimension, first we consider a square. We see that if we break the square into pieces that are 1/n the size of the original square, we need n^2 pieces to reassemble the square. The fractal dimension of a square is (Devaney 189):

$$D = \frac{\log n^2}{\log n} = \frac{2\log n}{\log n} = 2.$$

We see that the topological and fractal dimensions of the square are equal.

The Cantor set has a well-defined fractal dimension. The Cantor set has 2^n intervals and a magnification factor of 3^n at any stage, so the fractal dimension of C is (Devaney 190):

$$D = \frac{\log 2^n}{\log 3^n} = \frac{n\log 2}{n\log 3} = 0.6309...$$

As we predicted, the dimension of the Cantor set is between 0 and 1. The Cantor set does not have topological dimension, but it does have a well-defined fractal dimension. This shows that the Cantor set is indeed a fractal.

4 Fractals in Art

Benoit Mandelbrot considered his fractal geometry to be a new form of art (Mandelbrot 21). He claims fractal geometry as an “art for the sake of science,” and refers to the fractal as a useful beauty (Mandelbrot 22). Art historians and mathematicians, such as Mandelbrot, have been pondering the connections between the fields of art and mathematics for decades. Here, we will connect the Cantor set to art and architecture.

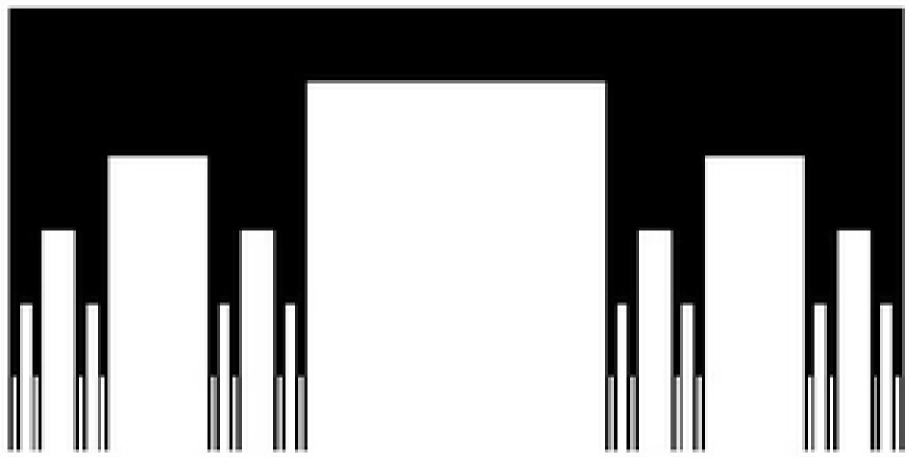

Figure 5: Connected Cantor Set, (Tex Stack Exchange).

Mandelbrot finds the coexistence of order and chaos in the issue of dynamics of iteration beautiful in itself (Mandelbrot 23). He also finds images of fractals artistic. In many cases, the traditional image of a common fractal is altered to make it more aesthetically pleasing. For example, the representation of the Cantor set above connects each iteration to the previous iteration. This Connected Cantor set (Figure 5) is more artistic than the usual representation of the set.

This image still represents the Cantor Set. Instead the of the traditional representation that consists of a set of separated lines, this representation exhibits one continuous object. It is a more organic image, which makes it more aesthetically pleasing. We can find examples in Chinese art and Architecture that resemble both this Connected Cantor set, as well as the traditional representation of the Cantor set.

4.1 Chinese Art

Fractals appear in many pieces of Chinese art. We can even find resemblance to the Cantor set, particularly the Connected Cantor set. Mandelbrot claims that fractals can serve as representations for natural objects (Mandelbrot 22), and we will apply this idea to Chinese art.

We first turn to the work of Guo Xi (1020-1090), a Chinese artist of the Northern Song dynasty (Bentley). Guo Xi painted in the black and white monumental landscape rugged style (Murashige 343). The rugged monumental landscape style originated in the previous Five Dynasties period and was initiated by painter Li Cheng (Bentley). It featured "crab-claw," defoliated tree branches. During the Northern Song period, Guo Xi adapted this monumental style, accentuating the crab-claw branches (Bentley). Though Guo's work came long before the Cantor set was discovered, we can find a resemblance to the set in his art.

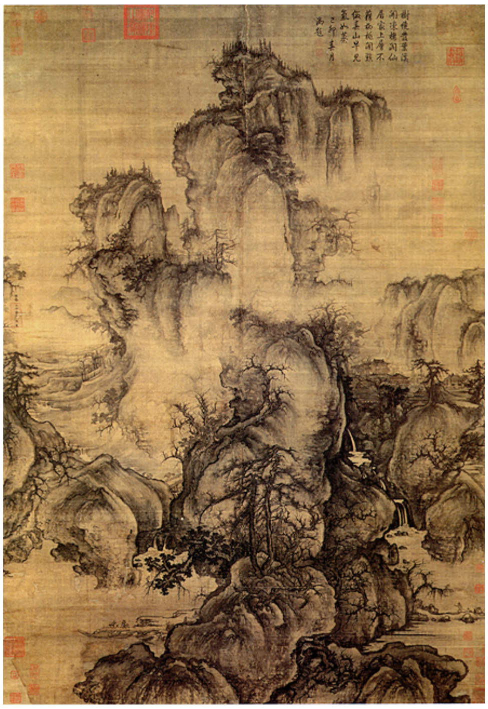

Figure 6: *Early Spring*, Guo Xi, ink on paper; 1072.

We consider *Early Spring*, one of Guo Xi's most famous works (Figure 6). This piece, painted in 1072, features the twisting crab-claw branches that the artist was known for (Murashige 343). The branches begin with a thick branch size, and a smaller arm branches off from each larger branch. This process is repeated on each smaller branch until the brush stroke becomes too thin to possibly be drawn. This process is reminiscent of the iterative process we use to construct the Cantor set. These branches also resemble our representation of the Connected Cantor set in Figure 5. This Cantor set representation shows each iteration connected to the next iteration in a branch-like way. The trees in Guo Xi's *Early Spring* resemble a version of our connected Cantor set in which the branches have been turned and twisted in different directions.

Guo Xi worked in a time where Song neo-Confucianism was the most prominent philosophy accepted by the people of China. This philosophy influenced both the subject matter and style of the work at the time (Bentley). A major concept explored in this type of neo-Confucianism is *li*, which means "inner structure." There are three different levels of *li*: the human level, the natural level, and the heavenly level (Li, Yan 205). The goal of each person is to align her own moral inner structure, which has been corrupted by emotions, with those of nature and heaven (Bentley). These philosophical levels are fractal-like. The ultimate goal would be for the *li* to be "self-similar" at each level. The human at the first philosophical level would like to make her inner structure "look" like the the inner structure of the next two levels. No matter the level, *li* should look the same. In other words, *li* should be self-similar. The concept of *li* is defined in a similar way to the way we define a fractal. Thus, even the philosophy behind Guo Xi's *Early Spring* resembles a fractal structure.

Figure 7: *Seven Junipers*, Wen Zhengming; Ming dynasty.

We will also consider Wen Zhengming (1470-1559), a Chinese scholar and painter from the Ming dynasty (Bentley). His famous *Seven Junipers* features twisted Juniper tree branches (Figure 7). These branches share the same resemblance to the Connected Cantor set representation in Figure 5 as those of Guo Xi's trees. This is an even more distorted version of our Connected Cantor set, but it still exhibits the thinning out effect we observed in *Early Spring*. By examining the works of Guo Xi and Wen Zhengming, we find a resemblance to the Cantor set. We see that the Cantor set, and fractal structure in general, can be applied in the context of Chinese art.

4.2 Architecture

Fractals also appear in architecture. We can find the Cantor set in the patterning of windows or other features on buildings. For example, we look to the AT&T building, now known as the Sony Tower, in New York City (Figure 8). The building was completed in 1984 and was designed by architect Philip Johnson and his partner John Burgee. To find a Cantor set, we consider the pattern of the windows on the front face of the building.

The top level of windows is in a symmetrical pattern. From the left, there is one medium-width window, then three large-width windows. The central section of windows contains eight window sections with small widths. The windows on the right side of the central section mimic the pattern of those on the left side. We will consider the windows themselves as part of our Cantor set and the concrete as the part we remove. At the next level of windows, each large-width window is "split" into four windows. At this iteration of the set, more points, represented by the concrete, have been removed. We can think of the pattern of the windows as a Cantor set. This is an example of a Cantor set that is not the traditional middle-thirds set.

Figure 8: Sony Tower, Philip Johnson and John Burgee, New York City, New York; 1984.

For another depiction of the Cantor set in architecture, we turn to a much older example. We can find the Cantor set in the capitals of Egyptian columns. For example, consider this column capital from the Temple of Dendur from 15 BC, which now resides in the Metropolitan Museum of Art in New York City (Figure 9).

Figure 9: Column Capital from Temple of Dendur, Metropolitan Museum of Art, 15BC.

The capital features bundles of papyrus stalks and lotus leaves, which take the form of a curve. The top curves are split into two smaller curves by removing a center section. The two smaller curves also shift away from the center of the larger curve. This process is repeated three times on this particular capital. This capital resembles a Cantor set in that various intervals of marble are removed through an iterative process. The Egyptians may have even intentionally used an iterative process to create this motif.

5 Conclusion

Georg Cantor's classical middle-thirds set exhibits intriguing mathematical properties. We showed that the Cantor set is uncountable, which is surprising because it seems that it should be a small set. We also proved the non-intuitive quality that though the Cantor set contains no subintervals, it also contains no isolated points. We can produce the Cantor set through a two-dimensional system of two functions. This study revealed the Cantor set as an attractor to an iterated function system. We also considered the Cantor set as a one-dimensional system of points that are not sent to infinity through exploration of the Tent Function. Discussion of the Cantor set as a fractal led us to find that C has a fractal dimension between 0 and 1.

Benoit Mandelbrot considered his fractal geometry an art form. We considered the Cantor set as an artistic form, with a focus in two different areas. Our Connected Cantor set representation resembles the trees in works by Chinese painters Guo Xi and Wen Zhengming. We also found a resemblance to this Connected Cantor set in the Song Tower of New York City. Lastly, we considered the column capital of the Egyptian Temple of Dendur and found a more classic representation of the Cantor set.

We have taken a complex mathematical set and applied it to the world of art. The Cantor set not only proves to be a set with interesting mathematical properties, but also a beautiful mathematical object with multiple applications in an artistic context.

Biblography

April 11, 2015

References

"Drawing Cantor Set." Stack Exchange. 25 Feb. 2012. Web. 7 Apr. 2015. http://tex.stackexchange.com/questions/31999/drawing-cantor-set.

Bentley, Tamara. Personal Communication, November 2014.

Dauben, Joseph Warren. *Georg Cantor: His Mathematics and Philosophy of the Infinite*. Cambridge: Harvard University Press, 1979. Print.

Devaney, Robert L. *A First Course in Chaotic Dynamical Systems: Theory and Experiment*. Perseus Books Publishing, LLC., 1992. Print.

Li, Jinglin, and Xin Yan. "The Ontologicalization of the Confucian Concept of 'Xin Xing:' Zhou Lianxi's Founding Contribution to the Song-Ming Neo-Confucianism." *Frontiers of Philosophy in China* 1.2 (2006): 204-21. JSTOR. Web. 7 Apr. 2015. http://www.jstor.org/stable/30209964.

Mandelbrot, Benoit B. "Fractals and an Art for the Sake of Science." *Leonardo Supplemental Issue* 2 (1989): 21-24. JSTOR. Web. 7 Apr. 2015. http://www.jstor.org/stable/1557938.

Murshige, Stanley. "Rhythm, Order, Change, and Nature in Guo Xi's Early Spring." *Monumenta Serica* 43 (1995): 337-64. JSTOR. Web. 7 Apr. 2015. http://www.jstor.org/stable/40727070.

Peak, David and Michael Frame. *Chaos Under Control: The Art and Science of Complexity*. W.H. Freeman, 1992. Print.

Ross, Kenneth A. *Elementary Analysis: The Theory of Calculus.* Springs Science+Business Media, 2013. Print.

Willard, Steven. *General Topology.* Reading: Addison-Wesley Publishing Company, 1970. Print.