Stochastic Variation of Driver Behavior Characteristics in Microscopic Traffic Flow Modeling

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Abstract

Modeling traffic flow has historically been tackled using a variety of different approaches. The microscopic method uses a system of ordinary differential equations to model the interaction of a single car with its surroundings. This model has fixed parameter values for all driver behavior characteristics, such as reaction time and reaction strength. Given that these parameters represent driver behavior, it is more realistic to introduce some stochasticity into the model parameters and observe the effect on the resulting traffic conditions. In this paper, direct simulation results show that stochastically varied parameter values produce larger free-flow regions, faster stabilization to free-flow conditions, and reduced traffic intensity overall. Fundamental diagrams show that the model behaves as predicted by real traffic data.

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1 Introduction

Traffic flow modeling has long been at the forefront of applied mathematics and civil engineering. How can we make transportation efficient for a large number of people? As populations climb all around the world, answering this question could save large amounts of time, money, and even lives. Many different mathematical models exist that attempt to describe and predict the behavior of vehicles, traffic, and traffic jams. Among the most widely used types of traffic models are microscopic and macroscopic models. Macroscopic modeling involves the use of partial differential equations to describe the density and velocity of cars along a road, similar to water flowing through a pipe. A more comprehensive study of macroscopic traffic modeling can be found in [7]. The microscopic model, with which we will be working in detail, deals with the interaction between a single car and its surroundings. Essentially, the behavior of traffic depends on the distance between each car and the next, and the driver's reaction to changes happening in front of them. Most microscopic models use a set of ordinary differential equations to describe these interactions for each vehicle. For further reading, Chapter 7 in [5] provides a detailed background on the different models of traffic flow theory.

The microscopic model has several advantages over the macroscopic. For one, the micro model contains more detail because it describes positions and behaviors of individual cars along a road. This allows us to make calculations of traffic flow/flux and road density, which are the main components of macroscopic models. In addition, we can account for different types of driver behaviors in various traffic situations, which permits us to introduce stochasticity, or randomness, into our model. Effects of stochasticity will be further discussed later in the paper. The specificity of the micro model comes at the expense of simplicity in the macroscopic model. Calculations in the macroscopic model are simpler because they do not have to account for and keep track of many vehicles at a time. For our purposes, we will accept more complex calculations in order to investigate the effects of realistic driver characteristics on the model.

The model we will be working with is an optimal-velocity model - a system of two ordinary differential equations that determine each car's position and optimal following distance (the distance between them and the car in front of them). A team of researchers at Brown University and the Technical University of Denmark created this model based on an existing optimal-velocity model presented in [1], in order to investigate the effects of relative car velocity on traffic behavior. Inherently, this model involves choosing parameters to represent each driver's reaction time and reaction "strength" to changes in velocity of the preceding car. Since they are indeed parameters, this model does not permit any variability between individual drivers, which is a very unrealistic condition to assume. Various works, such as [4] and [6], deal with stochastic variation in the reaction time to changes in *position* of the preceding car, however they do not include velocity-dependent parameters. Results from these papers are not conclusive about the effects of stochasticity.

The goal of this senior thesis is to explore the stability of traffic jam formations when the parameters are more varied. Do the solutions to system become unstable with a non-homogeneous population, or does free-flow prevail? First we investigate the effects of each parameter on the model. Then we introduce stochasticity to simulate a random sample of driver behaviors in the population and observe the outcome. Finally, we discuss fundamental diagrams, a major tool used in all types of traffic flow modeling, to assess the accuracy of our model.

2 Model Derivation

When developing a microscopic model for traffic flow, it is important to consider the main factors that each individual driver may experience. In reality, a driver may be aware of three or four individual cars in front of them, as well as a few cars behind them. For simplicity, we restrict these factors to the behavior of just the preceding car, specifically the velocity and relative position. Since the goal is to produce realistic driver behaviors with our model, we must consider what these behaviors might be and how to express them using variables and parameters.

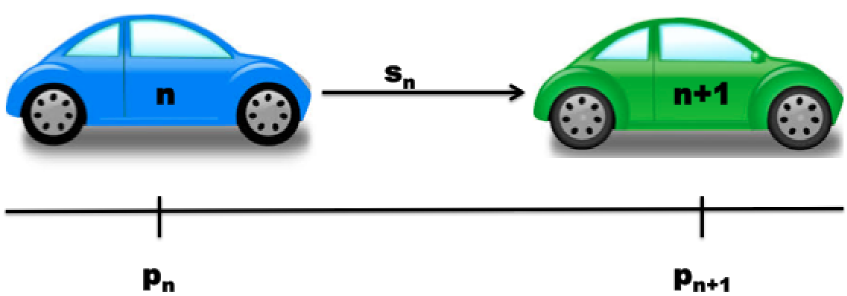


Figure 1: A diagram of the basic ideas behind the car-following microscopic traffic model.

2.1 The Model

Consider Car n traveling along a road, following Car $n+1$ at a moderate speed, as illustrated in Figure 1. If Car n and Car $n+1$ are too close together, irrespective of any other traffic, Car n will adjust its acceleration (slow down) in order to maintain what it considers a safe following distance, which we will call *headway*. That is, a distance long enough so that it would have enough time to react to a sudden stop from Car $n+1$, begin to brake, and come to a stop, without a collision. So we know that acceleration will depend on the reaction time of the driver, the relative positions of Car n and Car $n+1$ (or the distance between two cars), the velocity of Car n, and some desired individual headway distance. The basic model that describes this behavior is called an Optimal-Velocity Model (OVM) and is as follows:

$$\delta \ddot{p}_n = \mathcal{V}(p_{n+1} - p_n - \bar{s}) + v_0 - \dot{p}_n$$

where p_n is the position of the nth car, δ is the reaction time of the driver, v_0 is the optimal velocity (speed limit), and $\bar{s}$ is the optimal headway (road length divided by number of cars). We will use the conventional dot notation to represent the time derivative. This differential equation adjusts the driver's acceleration $\ddot{p}_n$ based on how close its current velocity is to an optimal velocity. The headway $p_{n+1} - p_n$ is compared with the optimal headway $\bar{s}$ and then adjusted by $\mathcal{V}$, a hyperbolic tangent function which acts to scale this difference to a number between -1 and 1. Therefore, the acceleration of each driver is dependent on the difference between the driver's current speed and the optimal velocity, and the difference between the current headway from the optimal.

Here is an example scenario that will help to understand the function of the OVM: If car n is traveling at the optimal velocity ($v_0 = \dot{p}_n$), but is too close to the car in front ($p_{n+1} - p_n < s_n$), then $V(p_{n+1} - p_n - s_n)$ will be negative and $v_0 - \dot{p}_n$ will be zero. Thus, $\ddot{p}_n$ will be negative, and car n will decelerate to match the target headway.

We also know, from basic practices of driving, that if Car $n+1$ accelerates to a greater velocity, Car n will try to reduce its headway and catch up. However, the OVM does not account for relative velocities and driver dependent headways. So we introduce a new variable s_n to represent each driver's individual desired headway. This headway must depend on relative car velocities, a certain optimal headway for the road conditions ($\bar{s}$), and the driver's personal reaction to changes in velocity.

With this additional variable, we arrive at the following non-dimensional model set forth in [2]:

$$\begin{aligned} \delta \ddot{p}_n &= \mathcal{V}(p_{n+1} - p_n - s_n) + v_0 - \dot{p}_n \qquad (1) \\ \alpha \dot{s}_n &= \bar{s} - s_n - \beta(\dot{p}_{n+1} - \dot{p}_n) \qquad (2) \end{aligned}$$

where we have replaced $\bar{s}$ in the OVM with s_n, determined by Equation (2), and introduced two new parameters, α and β, which will be discussed in the following section.

The new addition, Equation (2), adjusts the individual's desired headway distance depending on two factors. First, the $\beta(\dot{p}_{n+1} - \dot{p}_n)$ term allows the car to increase its headway distance if the car in front is going slower than it, and decrease the headway if the car in front is going faster. Second, the $\bar{s} - s_n$ describes the tendency for cars traveling too close to the car in front to reduce its headway distance. Similarly, cars traveling at a distance greater than the optimal headway distance $\bar{s}$ will try to decrease their headway.

2.2 Analyzing the Model

We will be using a MATLAB program to simulate traffic of N cars driving on a circular road of length L, see Figure 2. The actions of the first car on the road are determined by the position and velocity of the last car (this eliminates the need for boundary values). Free-flow is defined as the situation in which cars move with uniform speed and headway distances. Free-flow solutions to the ODE model arise depending on the chosen parameter values. Traveling wave solutions correspond to traffic jams, which travel in the opposite direction of the traffic flow. Multiple simultaneous traffic jams are also possible. As we will see, certain (α, β) pairs produce stable free-flow conditions. Some create stable 1-, 2-, 3-, 4-, 5-, and 6-jam solutions (we will refer to these stable solutions as k-jam solutions). Still other values create unstable traffic jams.

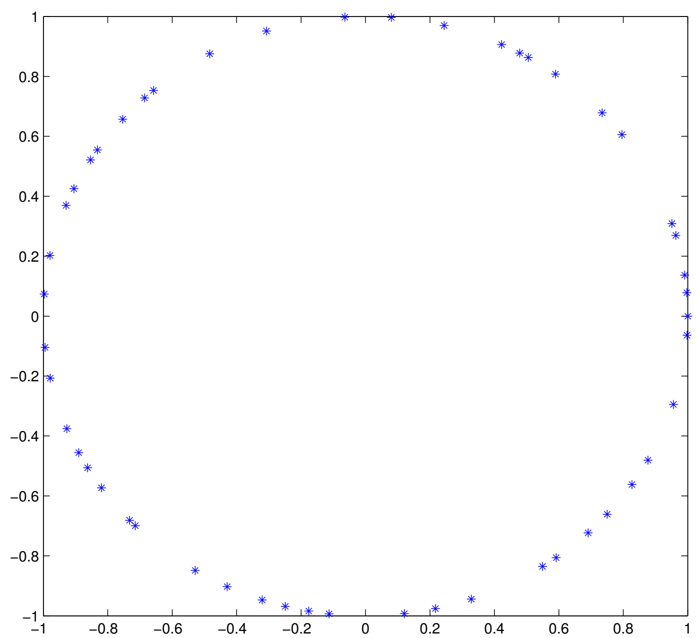


Figure 2: MATLAB direct simulation on a ring road of length L, with N cars placed randomly along the road.

We can visualize the effects of parameter alterations using free-flow stability diagrams (see Figure 3). A MATLAB program solves a related system of first-order ODEs for various (α, β) pairs. It then plots a graph at points where a complex conjugate pair of eigenvalues for this system crosses the imaginary axis, which denotes where the free-flow bifurcates into traffic jams [2, p. 2].

In Figure 3, each curve divides the $\alpha\beta$ plane into two sections. For example, let's examine only the magenta curve, which represents $k = 1$, or a single traffic jam. The area bounded by the curve and vertical axis is where we see stable free-flow solutions to our model. To the right of the curve, we see 1-jam solutions, which has the shape of a single period of a sinusoidal waveform. When we overlay all curves for $k = 1$ to 6, we see that the stable free-flow region exists in the lower left corner on the positive $\alpha\beta$ axis. Outside this region, various different jam solutions are possible.

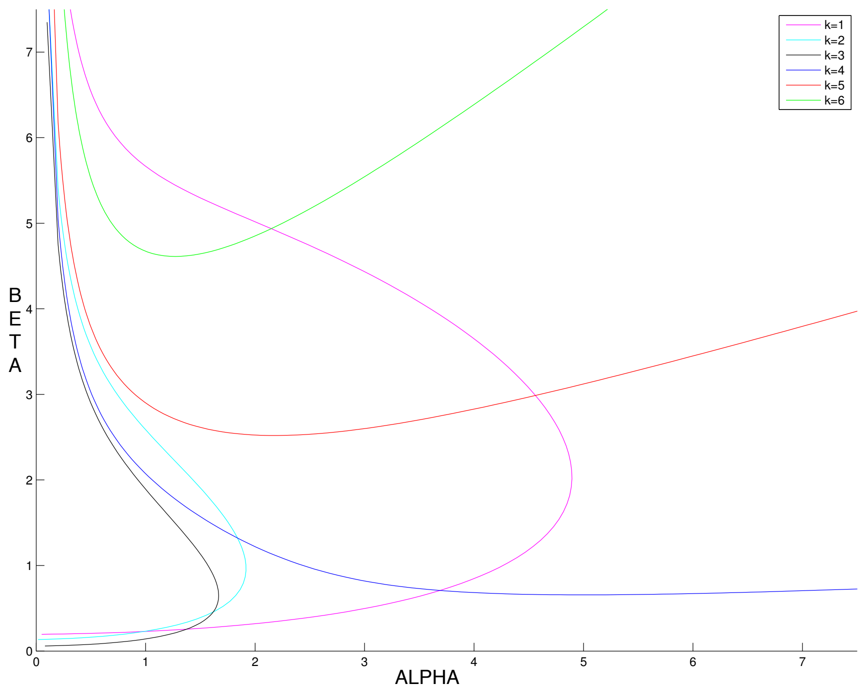


Figure 3: Free-flow stability diagram for a 20-car scenario.

3 Parameters

There are numerous important parameters in the model that we will be using to analyze traffic flow. Table 1 lists each parameter used by the model and briefly describes its function. The parameters v_0 and $\bar{s}$ are more closely related to the road conditions and are not dependent on individual driver behavior. In the next section we will explain how to choose realistic values for these parameters. For this project, the parameters that we are most interested in are β and α, for the reasons explained in Section 2.

Parameter	Description
δ	reaction time
v_0	optimal velocity
$\bar{s}$	optimal following distance (based on road length and number of cars)
α	safety distance adjustment time
β	strength of reaction to changes in relative velocity

Table 1: Parameters used in the model.

4 Incorporating Realistic Data

4.1 Transitions from Non-dimensional to Dimensional

Since the goal of this project is to realistically model traffic flow, every effort must be made to incorporate realistic parameter values and data. In order to find realistic values for velocity, optimal headway, reaction times, etc., it is important to dimensionalize the current model. As a part of my summer Research Experience for Undergraduates (REU) at Brown University, my team worked closely with these aspects of the model. We used the following length and time conversion based on a similar dimensionalization made in [3]:

$$x_n = \ell_0 \, p_n \tag{3}$$

$$\tau = t_0 \, t \tag{4}$$

where $\ell_0 = 11.63\text{m}$ and $t_0 = 0.69\text{s}$. The variables x_n and τ represent the dimensional version of the non-dimensional variables p_n and t. The length and time dimensions are now meters and seconds, respectively (another simple conversion gives miles and hours). The parameters $\tilde{\delta}$, $\tilde{\alpha}$, $\tilde{\beta}$ are the corresponding parameters in the dimensional model and have units of seconds. The parameters $\tilde{\delta}$ and $\tilde{\alpha}$ are reaction times and $\tilde{\beta}$, while also measured in seconds, behaves more like a reaction "strength". That is, a lower value of $\tilde{\beta}$ results in a moderate adjustment of the target headway, while a high $\tilde{\beta}$ causes the driver to aggressively adjust the target headway.

4.2 Japanese Data

In [3] a group of applied mathematicians collaborated with a transportation research group in Japan to obtain realistic parameter values for their model. The parameters were calculated using actual traffic data on Japanese motorways. Another data set in this paper contained values for city traffic in Stuttgart, Germany, which we excluded because the city traffic was too variable (stops, accelerations, turns, etc.) We converted these Japanese traffic values to our model and obtained $v_0 = 0.913$ and $\bar{s} = 2.15$. Using the dimensional conversions made in Section 4.1, this gives a maximum velocity of $v_0 = 37.6$ mph, and a optimal headway distance of $\bar{s} = 25$ meters. These values give us a better picture of what sort of traffic we will be dealing with in the following analysis.

5 Investigating Parameters of the Model

The goal for this section of the project is to observe how sensitive the model is to the parameters α and β. This portion of research was also conducted as part of the summer REU at Brown University. In the model, α and β appear in the second differential equation that describes how drivers react to changes in velocity of the car in front of them. As a reminder, α represents the time the driver takes to adjust his/her headway back to the optimal/desired headway s_n, after a change in velocity is observed. Similarly, β dictates how sensitive the driver is to changes in velocity. That is, a high value of β would represent a proactive driver who tends to make a large adjustment to his/her own headway when he/she detects even a slight change in the speed of the car in front.

From Section 2, we know there exists a region of α and β in which free-flow is stable. Figure 3 shows an example of the free-flow stability diagrams created using real-world traffic data on Japanese motorways taken from [3]. The small region in the bottom left corner on the graph (bounded by the black curve) is the region of (α, β) pairs that produce stable free-flow. Initially, the authors of [2] fixed the (α, β) values for every car on the road. Given that α and β represent driver behavior, it would be more realistic to vary these parameters for each car in the simulation.

5.1 β

First we examine the sensitivity of the β parameter, specifically the effects of altering the β-value for a single car. Using a MATLAB program, we can produce direct simulations of traffic and manipulate the parameters as desired. First, specific (α, β) points were chosen for which traffic jams were observed in the MATLAB direct simulation. These coordinates were identified using free-flow stability diagram (see Figure 3). Then all cars were fixed to these parameter values, while a single car, chosen at random, was significantly reduced ($\beta \approx 0.1$). Under these conditions, the direct simulations revealed that traffic jams were alleviated and the system stabilized to free-flow. Figure 4 gives a visual representation of these behaviors. Each plot shows the distances between the cars on the ring road. If a traffic jam solution is present, the plot would appear sinusoidal, with the number of peaks corresponding to the number of jams. The higher the amplitude of the sine wave, the heavier the jam. When the simulation begins, the cars are given initial positions placed randomly around the road.

From these observations, we propose that a few moderate drivers (in a sea of otherwise aggressive drivers), can alleviate traffic jams according to our model. That is, a few drivers with low β-values can dissipate highly jammed traffic and the resulting road condition is stable free-flow.

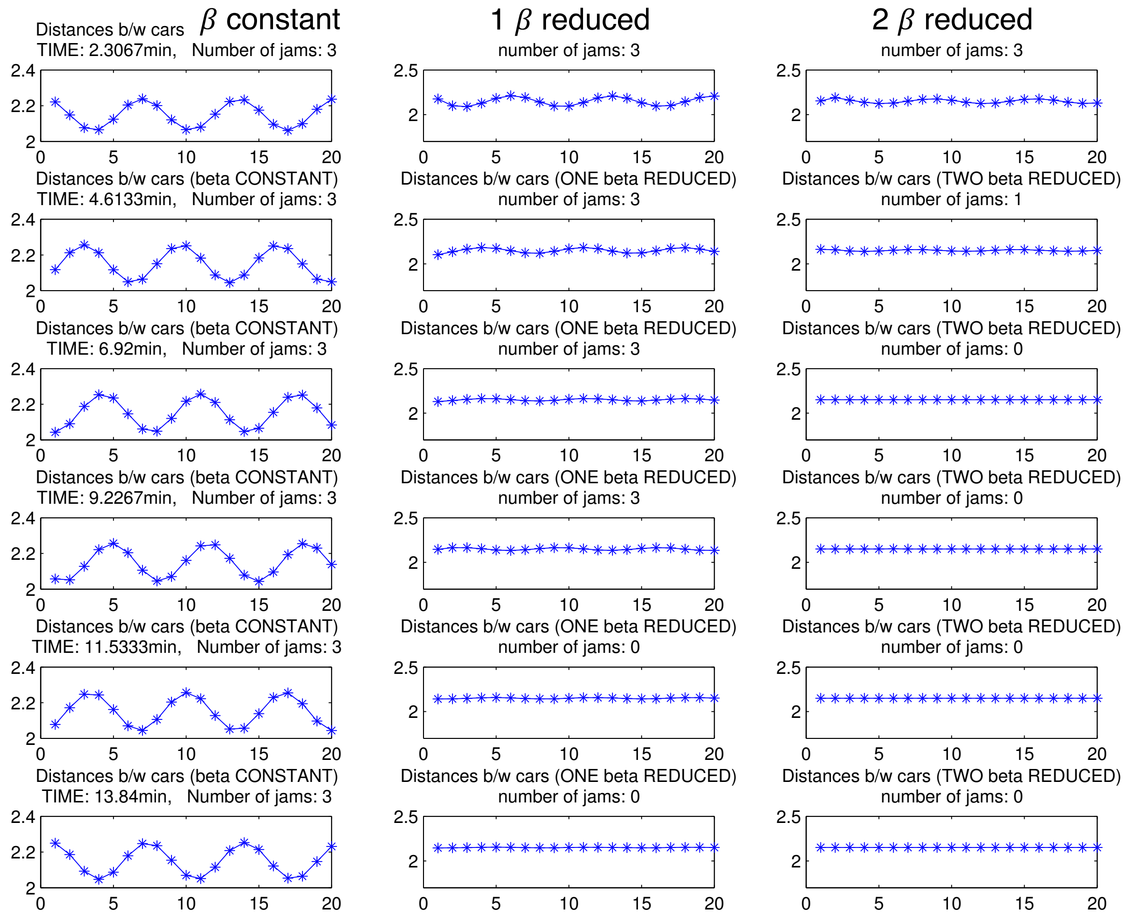


Figure 4: Direct simulation of model with 20 cars. First column represents fixed β for all cars. Second column represents one β reduced to 0.1. Third column represents two β reduced to 0.1.

5.2 α

We conducted a similar investigation for altering the α-values. (Reminder: the α parameter measures the reaction time of the driver). The model seems to be less sensitive to this parameter than to β. However, sufficiently reducing α for three or more drivers in a simulation of 50 cars will alleviate traffic jams in a similar fashion to the β-manipulation. Graphs similar to that in Figure 3 were produced for altering α-values. These results reinforce the notion that a few “good” drivers among a majority of drivers with (α, β) values in the traffic-jam region, can dissipate the traffic jams. By “good” drivers, we mean drivers that have quick reflexes (low α) and moderate reaction intensities (low β).

The reverse was also considered: do a few “bad” drivers destabilize free-flow? The answer is no. A system in stable free-flow is very unlikely to destabilize with the alteration of a few parameters.

6 Stochastic Variation of Driver Characteristics

With a better understanding of the model's parameter sensitivity and behavior, we now move to introduce some stochasticity, or randomness. This section largely details my personal research both at Brown, and continuing at Colorado College for my senior thesis. We would like to answer the following questions: Will stochastic variations in the parameters α and β impact the occurrence of traffic jams? Specifically, will a varied driver population increase or decrease the size of the free-flow stability region?

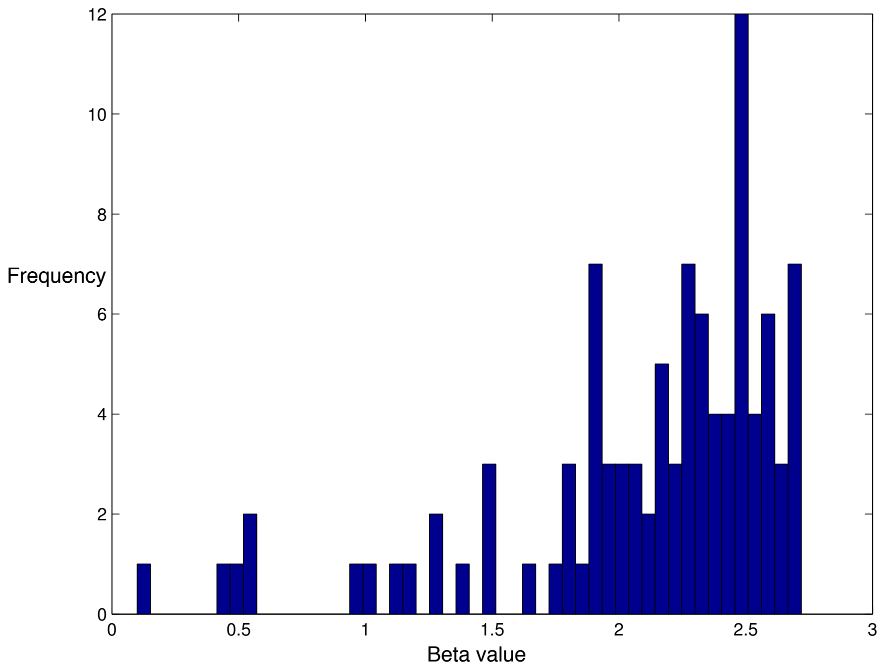


Figure 5: Exponential random distribution of beta values for 100 cars.

Section 5.1 hypothesized that samples containing a few good drivers can alleviate jams in otherwise jammed traffic conditions. In this case the frequency distribution of β-values would closely resemble an exponential random distribution as shown in Figure 5. Direct simulations showed that the system running under the exponential random distribution of β-values not only goes to free-flow, but stabilizes faster than an identical system (same initial conditions) with only one or two β-values reduced. See Figure 6 for reference.

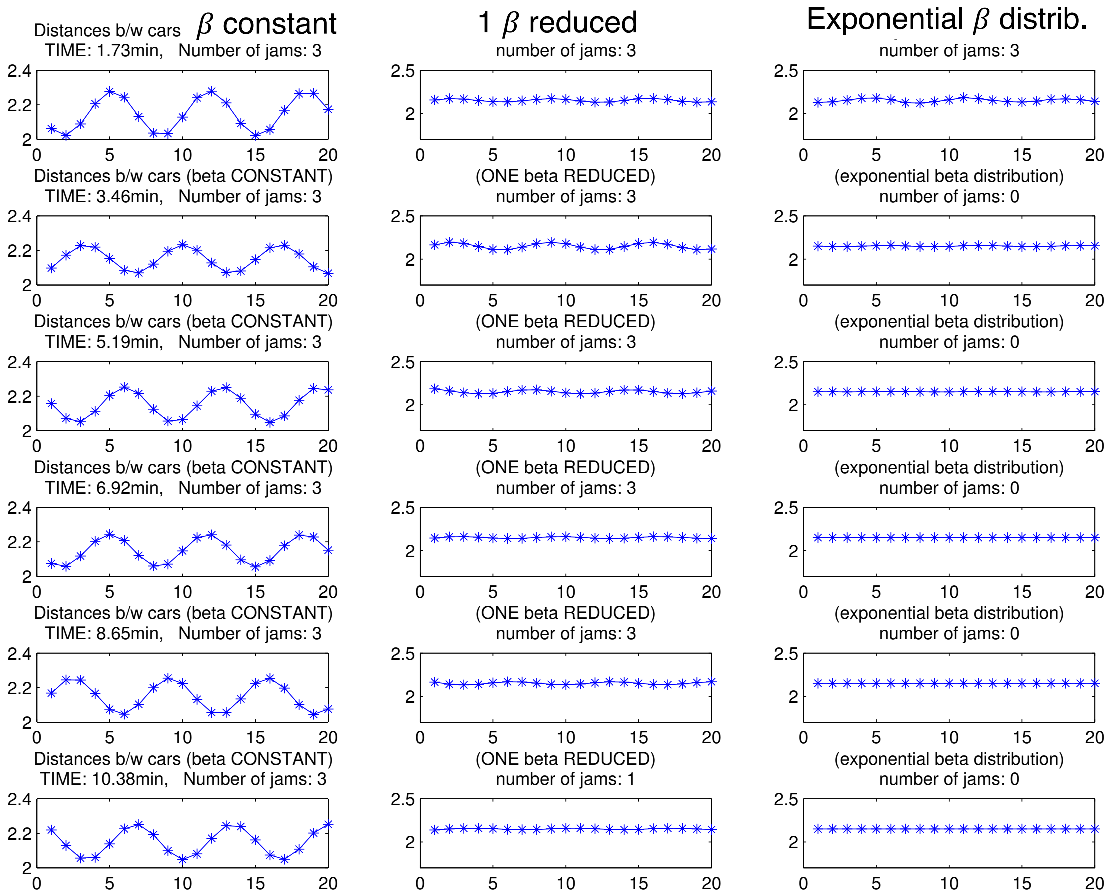


Figure 6: Direct simulation of model with 20 cars. First column represents fixed β for all cars. Second column represents one β reduced to 0.1. Third column represents an exponential distribution of β following the example in Figure 5.

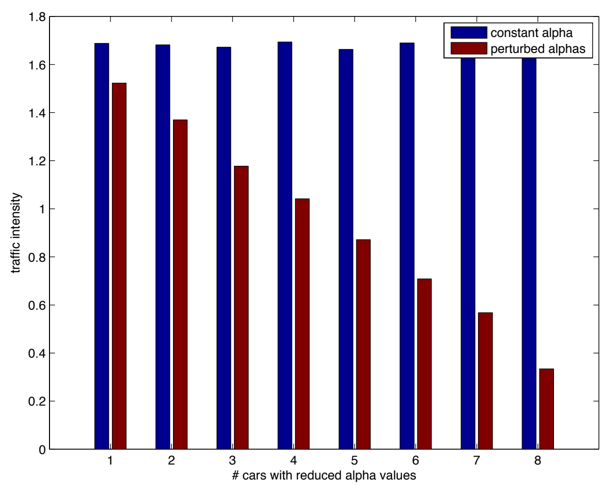
6.1 Traffic Intensity

In the MATLAB code, a jam is defined when the distances between cars is smaller than the optimal headway distance $\bar{s}$. Because of this, there is not much room for flexibility in counting the number of jams in a direct simulation. The distance between two cars may be only slightly less than $\bar{s}$ and a traffic jam would be detected, even if traffic appears to be flowing steadily to the human eye (for example, the middle diagram of last column in Figure 6). Therefore, in addition to the number of traffic jams observed, another method of quantifying traffic is needed. We (my summer research advisor, Bjorn Sandstede, and I) came up with the concept of intensity. Intensity, I, is calculated by centering the vector of distances between each car, $\vec{d}$, around zero, squaring it, and calculating the area under the resulting curve, as follows.

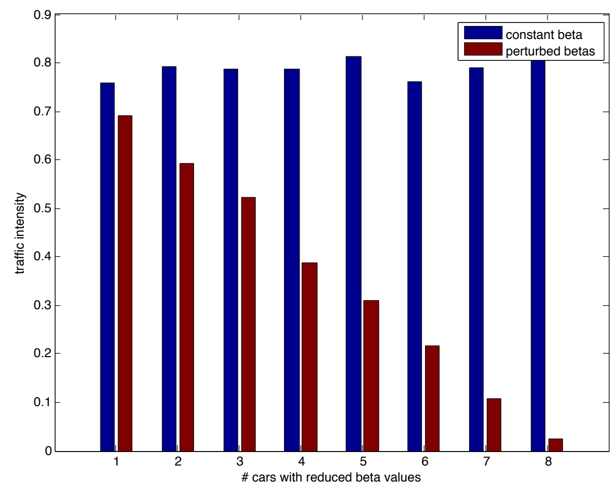
$$I \quad = \quad \int (\vec{d} - \bar{s})^2 \, \mathrm{d}x \qquad (5)$$

This accounts for the number of jams (number of peaks) as well as the severity of the jam (amplitude of the wave). This way, even if the number of traffic jams counted by

MATLAB remains the same, the intensity gives a clearer picture of what is happening on the road. As part of my research for this senior thesis, I used this measure to investigate threshold values for number of vehicles, n, with reduced parameter values that are needed to produce free-flow in an unstable jam environment. In Figure 7, we can see that traffic intensity drops visibly with each addition of a car with reduced parameter values.



(a) α

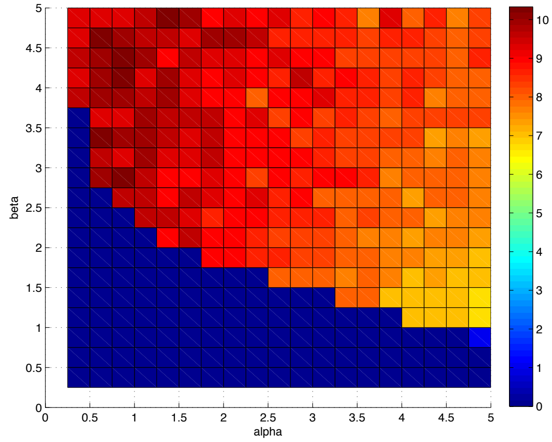


(b) β

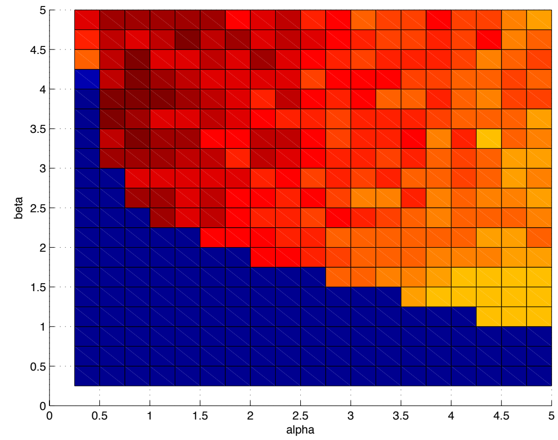
Figure 7: Direct simulation results for observing the effect of reduced α- and β-values on the traffic intensity.

6.2 Effects on the free-flow stability region

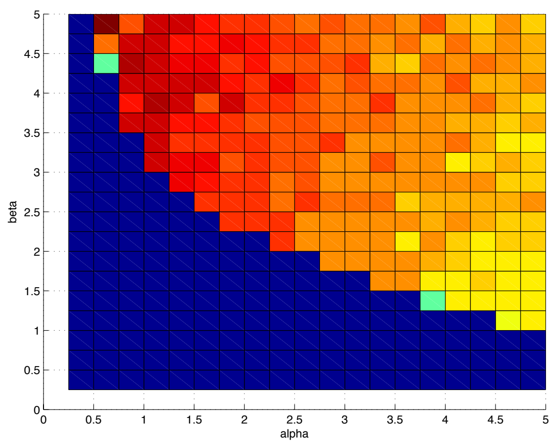
The previous arguments are sufficient in demonstrating the properties of the model, under specific conditions, i.e. specific (α, β) pairs. Recall that one of the questions we aim to answer deals with the effect of stochasticity on the size of the entire free-flow stability region. Also recall that the MATLAB program used to produce the free-flow stability diagrams is an ODE-solver, and therefore cannot include any variation in parameter values. We instead turn again to the direct simulations. I wrote a new MATLAB program that runs the model, with n randomly reduced β values, until a traffic condition stabilizes (free-flow or jammed) for every (α, β) coordinate from $\alpha, \beta = 0$ to 5 in steps of 0.25. Three trials are performed at each grid point in order to preserve accuracy of the test. The average number of traffic jams observed is then recorded in a matrix and plotted in a 3-D shaded surface plot, as shown in Figure 8. We display two-dimensional cross-sections of each plot for simplicity and readability. Plots are shown for various values of n, the number of cars with reduced β values.



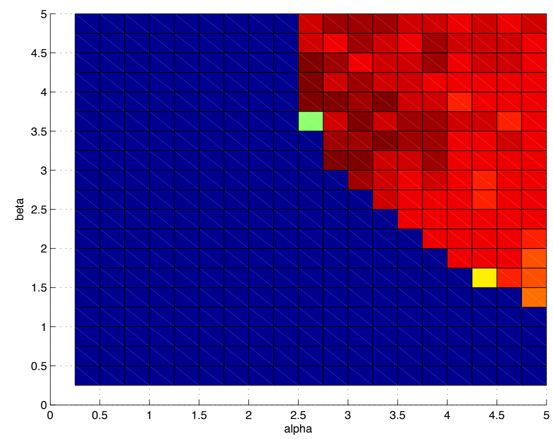
(a) All β constant ($n = 0$)



(b) $n = 1$



(c) $n = 5$



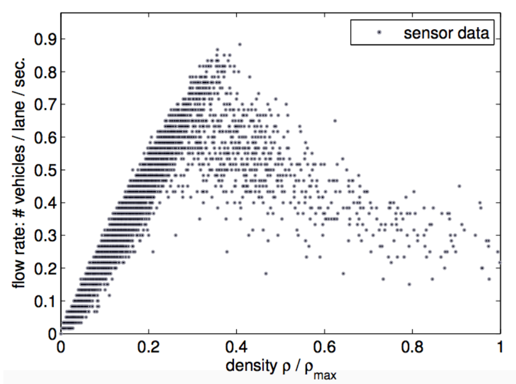
(d) $n = 15$

Figure 8: Investigating free-flow regions for varying n, the number of randomly reduced β-values. The sample size is 60 cars for this simulation.

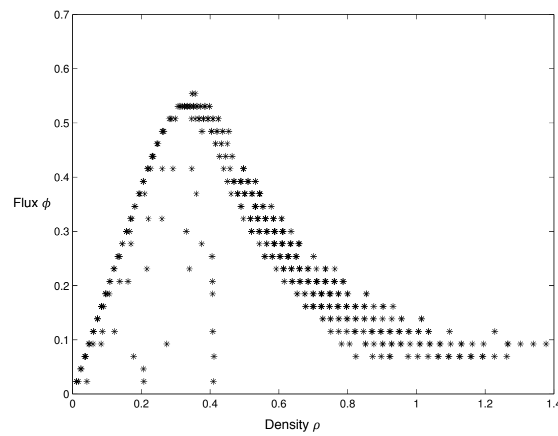
As the plots in Figure 8 show, reducing just one driver's reaction strength (β value) widens the entire free-flow region in α-β space. The more β-values that are randomly reduced, the wider the free-flow region becomes. These important results tell us that if we have a few more moderate drivers, the remaining population of drivers can have larger (α, β) values, *and remain in free-flow*, than they could before. That is, a varied driver population increases the probability for free-flow to occur.

7 Fundamental Diagrams

The fundamental diagram is a major tool used in all branches of traffic flow modeling. The diagram displays the relationship between flow or flux ϕ (the number of vehicles passing through a location per unit time, also equal to the product of speed and density), and density ρ (the number of vehicles on the road per unit distance). This is a useful tool when analyzing the capacity of a given road system. As shown in Figure 9, the flow/density fundamental diagrams consist of two major branches. The left branch is upward sloping and represents the free-flow conditions for the road. In this portion of the diagram, the velocity of cars increases as the density increases. Even though more cars are driving on the road, the cars are able to maintain or increase their velocity. At a certain point, however, this system breaks. At a critical density, the road becomes too packed for cars to be able to flow-freely and the flux drops. This critical density is located at the apex of the curve. Beyond the critical density, the diagram is downward sloping, implying that more densely packed roadways will have a lower flux, and therefore slower speed. In this portion of the graph, traffic becomes much less predictable; certain densities can produce a variety of different flow rates. This is where traffic jams occur.



(a) Fundamental diagram created using real traffic data. Reprinted from "Constructing Set-Valued Fundamental Diagrams from Jamiton Solutions in Second Order Traffic Models," by Semibold, Flynn, Kasimov, and Rosales, 2012, American Institute of Mathematical Sciences, 8, p. 747. Copyright 2013 by the AIMS.



(b) Fundamental diagram produced using model data, with Japanese motorways parameters, road length $L = 10,000$m, and $(\alpha, \beta) = (1, 2)$.

Figure 9: Two flow/density fundamental diagrams.

Figure 9a is a fundamental diagram that was produced using road sensors to record the flow rate and density of cars. Figure 9b is a fundamental diagram produced by running our ODE model in a MATLAB program that includes a road marker and counter to record the flux. As we can see, these two plots have the same general behavior as previously described. Most realistic models, both micro- and macroscopic, reproduce this general behavior. This is one of the reasons that fundamental diagrams are so widely used in traffic modeling; this phenomenon seems to occur quite often both in real world traffic and in model simulations. We accept the similarity between these two diagrams as reassurance that our model does a good job of accurately predicting the behavior of traffic in various conditions.

8 Conclusion

In this paper, we have discussed the detailed behaviors of a modified optimal-velocity microscopic traffic flow model. Building off of the work in [2], we have assessed the sensitivity of the model to the parameters α and β using repeated trials of a MATLAB direct simulation and incorporated real-world data from Japanese motorways from [3]. Direct simulation results show that stochastically varied parameter values give way to larger free-flow regions, faster stabilization to free-flow conditions, and reduced traffic intensity overall. Fundamental diagrams reassure us that the model behaves as predicted by real traffic data.

In the future, it would interesting to further investigate the α and δ parameters, as they also represent driver behavior characteristics (the MATLAB code that produces the diagrams in Figure 8 takes on average 15 hours to complete, so a detailed investigation of all parameters was not feasible for this senior thesis). Future studies could also explore sampling the parameters from a variety of different probability density functions and comparing the various outcomes. It would be interesting to investigate what sort of free-flow regions result in normally distributed parameter values for the driver population.

9 Acknowledgements

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References

[1] Bando, Hasebe, Nakayama, Shibata, and Sugiyama, *DYNAMICAL MODEL OF TRAFFIC CONGESTION AND NUMERICAL SIMULATION*, Phys. Rev. E, 51(1995), pp. 1035-1042.

[2] Carter, Christiansnsen, Gaididei, Gorria, Sandstede, Sorensen, and Starke, *MULTI-JAM SOLUTIONS IN TRAFFIC MODELS WITH VELOCITY-DEPENDENT DRIVER STRATEGIES*, SIAM J. Appl. Math., 60(2014), pp.1895-1918.

[3] Gaididei, Gorria, Berkemer, Kawamoto, Shiga, Christiansen, Sorensen, and Starke, *CONTROLLING TAFFIC JAMES BY TIME-MODULATING THE SAFETY DISTANCE*, Phys. Rev. E, (2013).

[4] Gasser, Seidel, Sirito, and Werner, *BIFURCATION ANALYSIS OF A CLASS OF 'CAR FOLLOWING' TRAFFIC MODELS II: VARIABLE REACTION TIMES AND AGGRESSIVE DRIVERS*, Bulletin of the Institute of Mathematics, 2(2007), pp.587-607.

[5] Hoogendoorn and Knoop, *TRAFFIC FLOW THEORY AND MODELING* in The Transport System and Transport Policy, Edward Elgar Publishing, 2013, pp. 125-159.

[6] Orosz, Krauskopf, Wilson, *TRAFFIC JAM DYNAMICS IN A CAR-FOLLOWING MODEL WITH REACTION-TIME DELAY AND STOCHASTICITY OF DRIVERS*, International Federation of Automatic Control Journals, 6(2006), pp. 199-204.

[7] M. Rascale, *AN IMPROVED MACROSCOPIC MODEL OF TRAFFIC FLOW: DERIVATION AND LINKS WITH THE LIGHTHILL-WHITHAM MODEL*, Mathematical and Computer Modelling, 35(2002), pp. 581-590.

[8] Seibold, Flynn, Kasimov, and Rosales, *CONSTRUCTING SET-VALUED FUNDAMENTAL DIAGRAMS FROM JAMITON SOLUTIONS IN SECOND ORDER TRAFFIC MODELS*, American Institute of Mathematics, 8(2013), pp.745-772.