Colorado College

Department of Mathematics

Markov Chains and Brownian Motion

Author:
Nicole Pey

Advisor:
Dr. Steven Janke

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1 Introduction

Consider a gambler's pot of money as she makes bets against the house, the ever-changing population of a certain bacteria, the disappearance of a given family name over time, or the fluctuating price of a stock. All of these systems vary randomly over time and have either a finite or infinite number of states, and are examples of stochastic processes.

Definition (stochastic process) The random variables, $\{X(t),\, t \in T\}$, defined on a common probability space, where T is a subset of $(\infty, -\infty)$ is known as a *stochastic process*.

Here T can be a subset of the integers or an interval of positive length. Additionally, our examples above all share the Markov property. This means that given the present state of the system, the past states have no influence on future states. The set of all possible states is called the state space.

Definition (Markov chain) A stochastic process $\{X_n : n = 0, 1, \dots\}$ with a finite or countably infinite state space S is said to be a *Markov chain*, if for all $x_i \in S$, and $n = 0, 1, 2, \dots,$

$$P(X_{n+1} = x_{n+1} \mid X_0 = x_0, \dots, X_n = x_n) = P(X_{n+1} = x_{n+1} \mid X_n = x_n).$$

In this paper, we will begin with an in-depth discussion of Markov chains and their various applications, and then we will focus on an important example, called the Random Walk. Once we have an understanding of a Random Walk in various dimensions, we will then arrive at Brownian motion by taking the limit of our simple Random Walk as the length of the step goes to zero.

1.1 Finite Chains

To get a precursory understanding of Markov chains we will begin with a finite example. Consider a Markov chain with state space $S = \{0, 1, 2, 3, 4, 5, 6\}$. The following is a transition matrix, which gives the probability of going from a given state $s \in S$ to any other state in S. Let's say our system is in state 0. With a probability of $\frac{1}{2}$, our system will stay at state 0 after one unit of time.

$$
\begin{array}{c c}
 & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} &
\begin{bmatrix}
\frac{1}{2} & 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\end{array}
$$

Now, let's take our system to be in state 1. With probability 1, our system will be at state 2 next. From state 2, our system will go to state 3 with probability 1. Finally, from state 3, the system will go to state 1 with probability 1. Once our system hits 1, 2, or 3, it is absorbed by $\{1, 2, 3\}$ and will not leave. We say the states $\{1, 2, 3\}$ and $\{4, 5, 6\}$ are recurrent and $\{0\}$ is transient. We will go into formal discussions of types of states and their properties in the next section.

2 Markov Chains

Markov chains have applications in physics, chemistry, statistics, biological modeling, finance, and elsewhere. Before going into a few in-depth examples, we must first understand the various properties of Markov chains.

2.1 Properties of States and Sets of States

States of a Markov chain are either recurrent or transient.

Definition (Recurrent) A state y such that a Markov chain starting at state y will return to y with probability 1 is called *recurrent*.

We can intuitively see that once a chain hits a recurrent state y, it will visit y infinitely many times. We will prove this theorem later.

Definition (Transient) A state y such that a Markov chain starting at state y will return to y with probability less than 1 is called *transient*.

It is more difficult to see, but a Markov chain will only hit transient states a finite number of times which, again, we will prove later. We can also classify groups of states in various ways.

Definition (Closed) A nonempty set C of states is *closed* if no state inside of C leads to any state outside of C.

In our finite example in 1.1, the set $\{1, 2, 3\}$ is closed. This set is also irreducible.

Definition (Irreducible) A closed set C is *irreducible* if the probability of going from x to y is greater than 0 for all choices of x and y in C.

The set $\{0, 1, 2, 3, 4, 5, 6\}$ is reducible.

2.2 Notation and Definitions

Now we will define the following:

$1_y(z)$	Function that equals 1 if $z = y$ and 0 if $z \neq y$.
$N(y) = \sum_{n=1}^{\infty} 1_y(X_n)$	The number of times that the chain is in state y.
T_A	Hitting time of A defined by $T_A = \min\ (n > 0 : X_n \in A)$.
$P^n(x, y) = P_x(X_n = y)$	Probability that a Markov chain starting at x will hit y in n steps.
$\rho_{xy} = P_x(T_y < \infty)$	Probability that a Markov chain starting at x will hit y in some positive time.
$G(x, y) = E_x(N(y))$	The expected number of visits to y for a Markov chain starting at x.

We must discuss $G(x,y)$ further. Consider

$$\begin{aligned} G(x,y) &= E_x(N(y)) \\ &= E_x\left(\sum_{n=1}^{\infty} 1_y(X_n)\right) \\ &= \sum_{n=1}^{\infty} E_x(1_y(X_n)) \\ &= \sum_{n=1}^{\infty} P_x(X_n = y) \\ &= \sum_{n=1}^{\infty} P^n(x,y). \end{aligned}$$

We will use this fact in later sections.

2.3 Theorem 1

(i) Let y be a transient state. Then

$$P_x(N(y) < \infty) = 1$$

and

$$G(x,y) = \frac{\rho_{xy}}{1-\rho_{yy}}, \ x \in S$$

which is finite for all $x \in S$.

(ii) Let y be a recurrent state. Then, $P_y(N(y) = \infty) = 1$ and $G(y,y) = \infty$. Also

$$P_x(N(y) = \infty) = P_x(T_y < \infty) = \rho_{xy}, \ x \in S.$$

If $\rho_{xy} = 0$, then $G(x,y) = 0$, while if $\rho_{xy} > 0$, then $G(x,y) = \infty$.

Proof Consider

$$\begin{aligned} P_x(N(y) \geq 2) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} P_x(T_y = m)P_y(T_y = n) \\ &= \left(\sum_{m=1}^{\infty} P_x(T_y = m)\right)\left(\sum_{n=1}^{\infty} P_y(T_y = n)\right) \\ &= \rho_{xy}\rho_{yy}. \end{aligned}$$

Similarly,

$$P_x(N(y) \geq m) = \rho_{xy}\rho_{yy}^{m-1}.$$

Let y be a transient state. So, $0 \leq \rho_{yy} < 1$ and thus,

$$\begin{aligned} P_x(N(y) = \infty) &= \lim_{m\to\infty} P_x(N(y) \geq m) \\ &= \lim_{m\to\infty} \rho_{xy}\rho_{yy}^{m-1} \\ &= 0. \end{aligned}$$

By the definition of expected value we have the following,

$$\begin{aligned} G(x,y) &= E_x(N(y)) \\ &= \sum_{m=1}^{\infty} mP_x(N(y) = m) \\ &= \sum_{m=1}^{\infty} m\rho_{xy}\rho_{yy}^{m-1}(1-\rho_{yy}). \end{aligned}$$

Intuitively, we can see where this final line comes from. The probability that the chain hits y exactly m times equals the probability that a chain starting at x will hit y once, then hit y exactly $(m-1)$ more times, and then never again. Looking at the final line a bit closer, and remembering that y is a transient state, so $\rho_{yy} < 1$, we see that $\sum m\rho_{yy}^{m-1}$ is the derivative of the geometric series. Thus, this sum is equal to the derivative of $\frac{1}{1-\rho_{yy}}$. Therefore we have,

$$G(x,y) = \frac{\rho_{xy}}{1-\rho_{yy}} < \infty.$$

Thus, (i) is proven.
Now, let y be a recurrent state so $\rho_{yy} = 1$. It follows that

$$\begin{aligned} P_x(N(y) = \infty) &= \lim_{m\to\infty} P_x(N(y) \geq m) \\ &= \lim_{m\to\infty} \rho_{xy}\rho_{yy}^{m-1} \\ &= \rho_{xy}. \end{aligned}$$

It follows that $P_y(N(y) = \infty) = 1$. Now consider,

$$\begin{aligned} G(y,y) &= E_y(N(y)) \\ &= \sum_{m=0}^{\infty} mP(N(y) = m) \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{m-1} P(N(y) = m) \\ &= \sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty} P(N(y) = m) \\ &= \sum_{n=0}^{\infty} P(N(y) > n) \\ &\geq \sum_{n=n_o}^{\infty} \epsilon \quad \text{(since } P_y(N(y) = \infty) = 1\text{)} \\ &= \infty \end{aligned}$$

Since $N(y)$ has a positive probability of being infinite, it has infinite expectation. Now, in the case that $\rho_{xy} = 0$, it is easy to see that $P^n(x,y) = 0$ and thus $G(x,y) = 0$. On the other hand, if $\rho_{xy} > 0$, then $P_x(N(y) = \infty) = \rho_{xy}$. Again, since $N(y)$ has a positive probability of being infinite, then $G(x,y) = \infty$.

□

This theorem gives us the fundamental difference between transient and recurrent states. If you hit a recurrent state y once, you will hit it infinitely often, but if the chain starts at some other state, it may be impossible to hit state y. If y is transient, no matter which state the chain starts in, you will hit y finitely many times.

2.4 Other Important Theorems and Properties

The following theorems give us some interesting results of Markov chains which we will not prove here.

- Let x be a recurrent state and suppose x leads to y. Then, y is recurrent and $\rho_{xy} = \rho_{yx} = 1$.

- A finite Markov chain must have at least one recurrent state.
- An irreducible chain has all recurrent or transient states.

2.4.1 Birth and Death Chain

Both queueing systems and populations are modeled by a birth and death chain. At each time step the population grows by one, lowers by one, or stays the same. We have the following transition function:

$$P(x,y) = \begin{cases} p_x, & y = x-1, \\ r_x, & y = x, \\ q_x, & y = x+1, \\ 0, & \text{otherwise.} \end{cases}$$

For a finite irreducible birth and death chain, all states are recurrent. Once two customers form a line at a grocery store, the number of people in line will return to two with probability one.
Now, let's say we're studying a species who's particular environment limits their population to d organisms and if they reach a population of 0, they are extinct. We call 0 an absorbing state. Once the population is 0, it will stay in that state forever. We can find the probability that the population will reach 0 starting at any population x.

Consider a lane in the grocery store the day before Super Bowl Sunday. Many customers have large cart loads. Let $p_x = \frac{1}{4}, r_x = \frac{3}{8}$, and $q_x = \frac{3}{8}$. With these probabilities, the line will keep growing. There are an infinite number of states in this case.

2.5 Stationary Distributions

In order to understand the behavior of Markov chains over time, we look to see if they have a stationary distribution. If $\pi(x)$, $x \in S$, are nonnegative numbers summing to one, and if

$$\sum_x \pi(x)P(x,y) = \pi(y),\ y \in S,$$

then π is called a stationary distribution.

Consider a frog sitting on lily pad 0 holding a bag of coins. Before he jumps randomly to another lily pad, he puts a coin down on lily pad 0. When he reaches the next lily pad he puts a coin down there as well. He continues to jump randomly from lily pad to lily pad leaving coins at each one for the remainder of the day. After his n jumps that day, there are piles of coins on each of the twenty lily pads in the pond. The amount of coins on a lily pad gives us the proportion of time the frog was on that lily pad out of the n jumps that he took. This illustrates how stationary distributions give us the proportion of time that the system will be in each state. Stationary means that if we start with a particular distribution, we will always have that distribution regardless of a shift in time.

2.5.1 Three state example

Consider a Markov chain having state space $\{0, 1, 2\}$ and transition matrix

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{bmatrix}$$

Now let's find the stationary distribution π. We have the following equations:

$$\frac{\pi(0)}{3} + \frac{\pi(1)}{4} + \frac{\pi(2)}{6} = \pi(0)$$
$$\frac{\pi(0)}{3} + \frac{\pi(1)}{2} + \frac{\pi(2)}{3} = \pi(1)$$
$$\frac{\pi(0)}{3} + \frac{\pi(1)}{4} + \frac{\pi(2)}{2} = \pi(2)$$

and

$$\pi(0) + \pi(1) + \pi(2) = 1$$

Thus, $\pi(0) = \frac{6}{25}$, $\pi(1) = \frac{2}{5}$, and $\pi(2) = \frac{9}{25}$.

This means that if we begin in state 0 with probability $\frac{6}{25}$, state 1 with probability $\frac{2}{5}$, and state 2 with probability $\frac{9}{25}$, then the next step will have the same distribution. Moreover, in this finite, irreducible case, we have a unique stationary distribution. No matter what initial distribution we start with, for large values of n, the distribution of X_n will be approximately equal to π.

2.5.2 Real World Example

Consider the states, Kansas and Missouri. There is a probability p that someone will move from Kansas to Missouri and a probability q that a person will move from Missouri to Kansas. We might want to know what the proportion of people in Missouri compared to the population of people in Missouri and Kansas over a long period of time and a stationary distribution would give us just that. In this model, we would assume that no one moved elsewhere or came from somewhere outside of these two states. Suppose $p = \frac{2}{3}$ and $q = \frac{1}{2}$, then we have the following transition matrix.

$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Thus, $\pi(0) = \frac{1}{3}\pi(0) + \frac{1}{2}\pi(1)$ and $\pi(1) = \frac{2}{3}\pi(0) + \frac{1}{2}\pi(1)$. Solving these equations we have a unique stationary distribution given by

$$\pi(0) = \tfrac{3}{7} \text{ and } \pi(1) = \tfrac{4}{7}.$$

This means that over time, $\frac{3}{7}$ of the population of Kansas and Missouri will live in Kansas and $\frac{4}{7}$ will live in Missouri.

2.5.3 Null Recurrent and Positive Recurrent States

It turns out that there are two types of recurrent states and stationary distributions are only interesting when the chain has *positive* recurrent states. First, we must define a few other things. We call m_x the mean return time to x as defined by $m_x = E_x(T_x)$ if m_x is finite and $m_x = \infty$ otherwise. Let

$$1_{\{T_x<\infty\}} = \begin{cases} 1, & T_x < \infty, \\ 0, & T_x = \infty. \end{cases}$$

This next function gives us the number of visits of a Markov chain to x during times $m = 1, \ldots, n$:

$$N_n(x) = \sum_{m=1}^{n} 1_x(X_m).$$

The following is the expected number of such visits:

$$G_n(z, x) = E_z(N_n(x)) = \sum_{m=1}^{n} P^m(z, x).$$

Notice that $\lim_{n\to\infty} \frac{N_n(x)}{n} = \frac{1_{\{T_x > \infty\}}}{m_x}$. Intuitively, this means that once a chain hits x, the amount of time that the chain will be in state x out of the first n units of time is about $\frac{1}{m_x}$ where m_x is the average amount of time between hitting x. If we take expectations, we have

$$\lim_{n\to\infty} \frac{G_n(z, x)}{n} = \frac{\rho_{zx}}{m_x}.$$

Definition (Null recurrent) A recurrent state y is *null recurrent* if $m_y = \infty$.

Thus, for a null recurrent state y,

$$\lim_{n\to\infty} \frac{G_n(x, y)}{n} = \lim_{n\to\infty} \frac{\sum_{m=1}^{n} P^m(x, y)}{n} = 0, \; x \in S.$$

Definition (Positive recurrent)A recurrent state y is *positive recurrent* if $m_y < \infty$.

Thus, for a positive recurrent state y,

$$\lim_{n\to\infty} \frac{G_n(y, y)}{n} = \frac{1}{m_y} > 0.$$

Let π be a stationary distribution. If x is a transient state or null recurrent state, then $\pi(x) = 0$.

2.5.4 Theorem 2

An irreducible positive recurrent Markov chain has a unique stationary distribution π, given by

$$\pi(x) = \frac{1}{m_x}, \; x \in S.$$

We must remember that a chain is irreducible if the probability of going from x to y is greater than 0 for all choices of x and y in S.

Proof Because our chain is positive recurrent, $\rho_{zx} = 1$. Thus,

$$\lim_{n\to\infty} \frac{G_n(z,x)}{n} = \frac{1}{m_x}. \tag{1}$$

From earlier, we know that $\sum_z \pi(z)P^m(z,x) = \pi(x)$. If we sum this equation on m and divide by n we have,

$$\sum_{m=1}^{n}\sum_z \pi(z)P^m(z,x) = \sum_z \pi(z)\frac{G_n(z,x)}{n} = \pi(x).$$

Thus, using the bounded convergence theorem, we have

$$\begin{aligned}\pi(x) &= \lim_{n\to\infty}\sum_z \pi(z)\frac{G_n(z,x)}{n}\\ &= \frac{1}{m_x}\sum_z \pi(z)\\ &= \frac{1}{m_x}\end{aligned}$$

If $\frac{1}{m_x}$ satisfies the following necessary properties of a stationary distribution, then it must be the unique stationary distribution:

$$\sum_x \frac{1}{m_x} = 1$$

and

$$\sum_x \frac{1}{m_x}P(x,y) = \frac{1}{m_y},\ y \in S.$$

To start, we know,

$$\sum_x P^m(z,x) = 1.$$

If we sum on m from 1 to n and divide by n we get,

$$\sum_x \frac{G_n(z,x)}{n} = 1. \tag{2}$$

Intuitively, it makes sense that

$$\sum_x P^m(z,x)P(x,y) = P^{m+1}(z,y),$$

since we can hit any x in m steps and then hit y in one step to take our chain from state z to y in $m+1$ steps. Again, if we sum this equation on m from 1 to n and divide by n we have

$$\sum_x \frac{G_n(z,x)}{n} P(x,y) = \frac{G_{n+1}(z,y))}{n} - \frac{P(z,y)}{n}$$

Taking the limit we have

$$\lim_{n\to\infty} \sum_x \frac{G_n(z,x)}{n} P(x,y) = \lim_{n\to\infty} \frac{G_{n+1}(z,y))}{n} - \frac{P(z,y)}{n}$$

so

$$\sum_x \frac{1}{m_x} P(x,y) = \frac{1}{m_y}.$$

If we sum equation (1) we have,

$$\sum_x \lim_{n\to\infty} \frac{G_n(z,x)}{n} = \sum_x \frac{1}{m_x}.$$

Flipping the order of the limit and sum and applying equation (2), we conclude

$$1 = \sum_x \frac{1}{m_x}$$

The infinite case is more difficult since we cannot swap the summations and limits. The idea of the proof is that we use a finite subset. Our equalities in the above equations become inequalities and we show that the strict inequality cannot hold. Thus, the two criteria are met.

□

2.5.5 Periodicity

Let x be a state of a Markov chain such that $P^n(x,x) > 0$ for some $n \geq 1$, i.e. such that $\rho_{xx} = P_x(T_x < \infty) > 0$. We define its period d_x by

$$d_x = \text{g.c.d.}\{n \geq 1 : P^n(x,x) > 0\}.$$

Then

$$1 \leq d_x \leq \min(n \geq 1 : P^n(x,x) > 0).$$

For example, states in an irreducible Markov chain have common period d. These chains are *periodic* with period d. If $d = 1$, we call the chain *aperiodic*. Knowing the period of a Markov chain gives us insight on how the chain behaves and, in particular, helps us determine the stationary distribution.

2.5.6 Theorem 3

Let X_n, $n \geq 0$, be an irreducible positive recurrent Markov chain having stationary distribution π. If the chain is aperiodic,

$$\lim_{n\to\infty} P^n(x,y) = \pi(y), \;\; x, y \in S.$$

If the chain is periodic with period d, then for each pair x, y of states in S there is an integer $r, 0 \leq r < d$, such that $P^n(x,y) = 0$ unless $n = md + r$ for some nonnegative integer m, and

$$\lim_{m\to\infty} P^{md+r}(x,y) = d\pi(y).$$

This theorem gives us the specific chains that converge to a unique stationary distribution no matter what distribution they start with. Think back to our three-state example from section 2.5.1. That chain is aperiodic!

3 Random Walk

We have seen Markov chains in action in a variety of contexts, but now we want to focus on the famous example of the random walk. Picture yourself standing on a number line at zero. You take a series of steps, whereby you only travel to adjacent integers on the line. From any point x on the number line, you will step to $x+1$ with a probability p and step to $x-1$ with probability $q = 1-p$. Here we have the transition function for this Markov chain. We must note that a random walk can have a finite or infinite state space and can be of any dimension. If we take infinitely many steps on the number line, it would be nice to know if we will come back to the origin infinitely many times or drift off in one direction. The following three sections will determine whether or not the origin is a recurrent state in symmetric random walks of various dimensions.

3.1 One-Dimensional

Let's look at the expected number of visits to 0 for a chain starting at 0 in the one-dimensional case. If we take one step away from the origin, we know we will have to take one step to get back. Our chain is periodic with a period, $d = 2$. Thus, we must take an even number of steps to return to 0. For our symmetric random walk, $q = p = \frac{1}{2}$.

$$\begin{aligned}
G(0,0) &= \sum_{n=1}^{\infty} P^{2n}(0,0) \\
&= \sum_{n=1}^{\infty} \binom{2n}{n} \cdot \left(\frac{1}{2}\right)^{2n} \\
&= \sum_{n=1}^{\infty} \frac{2n!}{n!n!} \cdot \left(\frac{1}{2}\right)^{2n} \\
&\approx \sum_{n=1}^{\infty} \frac{\sqrt{2\pi}e^{-2n}\sqrt{2n} \cdot (2n)^{2n}}{2\pi e^{-2n} n \cdot n^{2n}} \cdot \left(\frac{1}{2}\right)^{2n} \quad \text{by Stirling's Approx} \\
&= \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}}
\end{aligned}$$

By the comparison test, this series diverges. Since the ratio of the Stirling's approximation to $P^{2n}(0,0)$ goes to 1 as $n \to \infty$, $G(0,0)$ also diverges. Thus, the origin is a recurrent state. There is a positive probability of going from state 0 to any other state x, thus all states in a one-dimensional random walk are recurrent.

3.2 Two-Dimensional

Now let's consider the two-dimensional case. Picture yourself walking in downtown Colorado Springs. Each time you come to a street corner, you flip a coin to determine which of the four directions to go next. We would consider each block to be a step in our random walk. As with a one-dimensional random walk, it takes an even number of steps to get back to the origin and there is an equal probability of going in any of the four directions.

$$
\begin{aligned}
P^{2n}(0,0) &= \sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2n}{2k}\binom{2k}{k}\left(\frac{1}{4}\right)^{2n} \\
&= \sum_{k=0}^{n} \frac{(2n-2k)!}{(n-k)!(n-k)!}\cdot\frac{(2n)!}{(2n-2k)!(2k)!}\cdot\frac{(2k)!}{k!k!}\cdot\left(\frac{1}{4}\right)^{2n} \\
&= \sum_{k=0}^{n} \frac{(2n)!}{(n-k)!(n-k)!k!k!}\cdot\frac{(n!)^2}{(n!)^2}\cdot\left(\frac{1}{4}\right)^{2n} \\
&= \frac{(2n)!}{n!n!}\sum_{k=0}^{n} \frac{n!n!}{(n-k)!(n-k)!k!k!}\cdot\left(\frac{1}{4}\right)^{2n} \\
&= \binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}\cdot\left(\frac{1}{4}\right)^{2n} \\
&= \binom{2n}{n}\binom{2n}{n}\cdot\left(\frac{1}{4}\right)^{2n} \\
&= \frac{((2n)!)^2}{(n!)^4}\cdot\left(\frac{1}{4}\right)^{2n} \\
&\approx \frac{2\pi e^{-4n} n 2^{4n} n^{4n}}{4\pi^2 e^{-4n} n^2 n^{4n}}\cdot\left(\frac{1}{4}\right)^{2n} \qquad \text{by Stirling's Approximation} \\
&= \frac{1}{2\pi n}.
\end{aligned}
$$

In this calculation we used the following: $\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$.
We have $P^{2n}(0,0) \sim \frac{1}{n}$, so

$$
\begin{aligned}
G(0,0) &= \sum_{n=0}^{\infty} P^{2n}(0,0) \\
&= \sum_{n=0}^{\infty} \frac{1}{n} = \infty.
\end{aligned}
$$

The harmonic series diverges. By Theorem 1, the origin is a recurrent state for the two-dimensional random walk. Therefore, with the same reasoning as above, every state in a two-dimensional random walk is recurrent.

3.3 Three-Dimensional

As in lower dimensions, it takes an even number of steps to return to the origin. There is a probability of $\frac{1}{6}$ of going in any of the six directions.

$$
\begin{aligned}
G(0,0) &= \sum_{n=0}^{\infty} P^{2n}(0,0) \\
&= \sum_{n=1}^{\infty}\sum_{k=0}^{n}\sum_{q=0}^{n-k} \binom{2n}{2k}\binom{2k}{k}\binom{2n-2k}{2q}\binom{2q}{q}\binom{2n-2k-2q}{n-k-q}\cdot\left(\frac{1}{6}\right)^{2n} \\
&= \sum_{n=1}^{\infty}\sum_{k=0}^{n}\sum_{q=0}^{n-k} \frac{(2n)!}{k!k!q!q!(n-k-q)!(n-k-q)!}\cdot\left(\frac{1}{6}\right)^{2n} \\
&= \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{2n}\binom{2n}{n}\sum_{k=0}^{n}\sum_{q=0}^{n-k}\left(\frac{n!}{i!j!(n-i-j)!}\right)^{2}\cdot\left(\frac{1}{3}\right)^{2n} \qquad \text{after multiplying by } \frac{(n!)^2}{(n!)^2} \\
&\leq \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{2n}\binom{2n}{n}\left(\frac{1}{3}\right)^{n} m\sum_{k=0}^{n}\sum_{q=0}^{n-k}\frac{n!}{i!j!(n-i-j)!}\cdot\left(\frac{1}{3}\right)^{n}
\end{aligned}
$$

Here, m is the maximum of $\frac{n!}{i!j!(n-i-j)!}$. Note that the maximum occurs about when $i = j = \frac{n}{3}$ which can be shown carefully. We pull out the maximum and sum the rest, which ends up being equal to 1. As shown in section 3.1, $\left(\frac{1}{2}\right)^{2n}\cdot\binom{2n}{n} = \frac{1}{\sqrt{\pi n}}$.

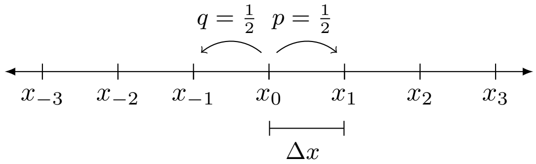
Thus,

$$\begin{aligned}
G(0,0) &\leq \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} \left(\frac{1}{3}\right)^{n} m \\
&\approx \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}} \left(\frac{1}{3}\right)^{n} \cdot \frac{\sqrt{2\pi} e^{-n} \sqrt{n} \cdot n^n}{(2\pi)^{\frac{3}{2}} e^{-n} (\frac{n}{3})^{\frac{3}{2}} (\frac{n}{3})^n} \text{ by Stirling's Approximation.} \\
&= \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}} \left(\frac{1}{3}\right)^{n} \cdot \frac{\sqrt{2\pi}\sqrt{n}}{(2\pi)^{\frac{3}{2}} (\frac{n}{3})^{\frac{3}{2}} (\frac{1}{3})^n} \\
&= \sum_{n=1}^{\infty} \cdot \frac{\sqrt{2}}{(2\pi)^{\frac{3}{2}} (\frac{n}{3})^{\frac{3}{2}}} \\
&= \sum_{n=1}^{\infty} \frac{c}{n^{\frac{3}{2}}}
\end{aligned}$$

This is a convergent series! Thus, we have a transient state! Although in one and two dimensions, the system returns to the origin with a probability of 1, that is not the case here. In three dimensions, there is a probability of .34 that you will return to the origin. Interestingly, random walks with dimensions *three or greater*, are transient Markov chains.

4 Brownian Motion

In 1827, botanist Robert Brown studied the motion of a particle of pollen suspended in water. It appeared that the particles moved randomly, undergoing a series of constant collisions. Albert Einstein, used the laws of physics to mathematically explain this movement in 1905. Then, in 1923, Norbert Weiner proved many results of this motion with mathematical rigor. To understand the random motion of particles that Brown noticed under his microscope, we will look at the random walk over continuous time. In particular, the limiting behavior of the random walk approximates the motion of a particle in each of the three directions.



4.1 Derivation

For a one-dimensional random walk, the position of the particle at time t is a random variable, X_t, which is the sum of independent, identically distributed random variables.

$$X_t = X_{\Delta t_1} + X_{\Delta t_2} + \cdots + X_{\Delta t_n} = \sum_{x=0}^{n} X_{\Delta t_x}.$$

Each $X_{\Delta t_i}$ has an expectation of

$$E(X_{\Delta t_i}) = (\Delta x)\frac{1}{2} + (-\Delta x)\frac{1}{2} = 0$$

and a variance of

$$Var(X_{\Delta t_i}) = E({X_{\Delta t_i}}^2) = (\Delta x)^2.$$

Thus,

$$EX_t = \sum_n \mathrm{E}X_{\Delta t_n} = 0$$

and

$$\begin{aligned}
VarX_t &= E({X_t}^2) \\
&= E(X_{\Delta t_1} + X_{\Delta t_2} + \cdots + X_{\Delta t_n})^2 \\
&= EX^2_{\Delta t_1} + \cdots EX^2_{\Delta t_n} + E(X_{\Delta t_i} X_{\Delta t_j}) \quad \text{where } i \neq j \\
&= EX^2_{\Delta t_1} + \cdots EX^2_{\Delta t_n} + EX_{\Delta t_i} EX_{\Delta t_j} \quad \text{since each } X_{\Delta t_i} \text{ is independent} \\
&= \left[\frac{t}{\Delta t}\right] (\Delta x)^2.
\end{aligned}$$

Here, $n = \left[\frac{t}{\Delta t}\right]$, where Δt is the length of the infinitesimal time increments and $\left[\frac{t}{\Delta t}\right]$ is the largest integer less than or equal to $\frac{t}{\Delta t}$.

Let $\Delta x = \sigma\sqrt{\Delta t}$; then $\Delta x \to 0$ as $\Delta t \to 0$. Therefore, Var $X_t = t\sigma^2$.

We get this definition of Δx from the process of diffusion. Albert Einstein developed the theory behind the irregular movement of particles when suspended in water, as influenced by dynamic equilibrium and the molecular-kinetic theory of heat. He found that the displacement of a particle is proportional to the square root of the change in time.

Another way to derive this mathematically is to begin with the assumption that X_t is continuous. Define $\varphi_1 = EX$ and $\varphi_2 = VarX$. We can show that $\varphi_1(t+\tau) = \varphi_1(t) + \varphi_1(\tau)$ and $\varphi_2(t+\tau) = \varphi_2(t) + \varphi_2(\tau)$. All continuous solutions to these functional equations are linear. It then follows that $EX_t = 0$ and $VarX_t = \sigma^2 t$.

4.2 Facts

- By the Central Limit Theorem, for large n, X_t has a normal distribution.
- As shown by physical observations, X_t has stationary, independent increments: $(X(t_0+t)-X(t_0))$. The distribution of these increments only depends on t.
- X_t is everywhere continuous but nowhere differentiable. This fact makes the graph of X_t difficult to study, as it requires the use of stochastic integration.
- Let $W(t)$ be a standard Brownian motion then $X(t) = \mu t + \sigma W(t)$, $t \geq 0$ is the Brownian motion with drift parameter μ. The concept of drift comes in handy when modeling the movement of a stock price where you need to consider inflation.

4.3 Application

Brownian motion can be used to model:

- The motion of a particle
- The variation in stock price over time
- The movement of a bacteria when a particular substance is introduced into their environment

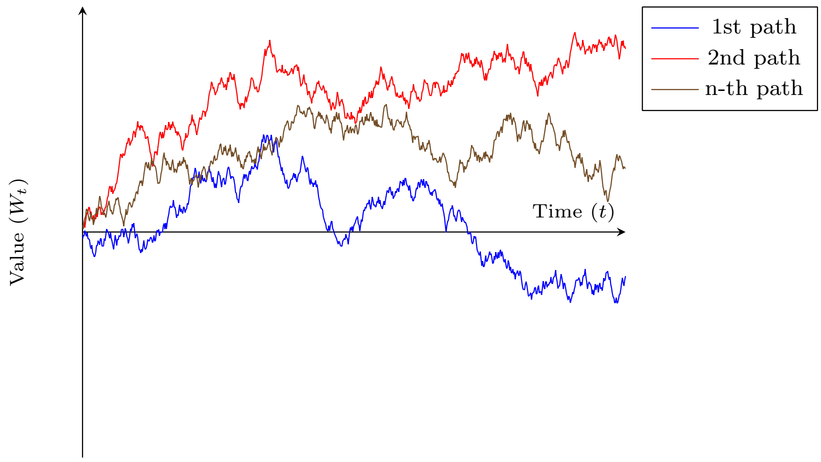


Figure 1: The continuous paths of a Brownian motion.

5 Conclusion

If a process is truly random how can we even begin to understand its behavior? As shown in this paper, we turn to probability theory to model stochastic processes and to answer important questions about how these systems work. We have looked at both finite and infinite Markov chains. Transition matrices help us to visually understand the structure of finite chains. Both for the finite and infinite case, studying $G(x, y)$, is critical to understand what's going on. This allowed us to understand how the random walk behaves in various dimensions by determining the recurrence of the origin. We have also seen how a limiting process of the random walk gives us something useful in physics, called Brownian motion. This application models biological and financial processes, two seemingly unrelated worlds.

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