Epidemiological Methods for Examining Bullying

A Thesis
Presented to
The Math and Computer Science Department
Colorado College
In Partial Fulfillment of the Requirements for the Degree
Bachelor of Arts

Kaitlyn Martinez
Faculty Advisor: Andrea Bruder

May 2015

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Abstract

Bullying is defined as a specific type of aggression, in which the behavior is intended to harm or disturb, the behavior occurs repeatedly, and there is an imbalance of power. This results in significant psychological damage in the victim, but also in the bully. Studies report the number of bullied children in middle schools as between 4% and 82%. The goals of our study are to understand how bullying behavior spreads in a population of adolescents, and to examine the impacts of the most common bullying intervention strategies. We propose a compartmental model parametrized using data on the prevalence of bullying. We compute the basic reproductive number R_0 and perform numerical simulations and a sensitivity analysis of the model. An extension of the simple model includes the most common intervention strategies. Numerical simulations suggest that the Traditional Disciplinary Approach, although commonly implemented, is the least effective of the intervention strategies we study.

1 Introduction

In a study of over 15,000 children in grades 6-10, it was found that nearly 30% of the children engaged in moderate or frequent bullying either as a bully, a victim, or both [11]. The numbers vary greatly from school to school, where in some schools bullying occurs rarely, and in others nearly 80% of the community is impacted [3]. Bullying is not characterized by mild bickering between children. The scientific community agrees that bullying is an entirely different behavior. They repeatedly define it "a specific type of aggression, in which (1) the behavior is intended to harm or disturb, (2) the behavior occurs repeatedly over time, and (3) there is an imbalance of power, with a more powerful person or group attacking a less powerful one" [11]. This can occur as physical attack, verbal harassment, social exclusion, spreading rumors, and cyberbullying [16].

The effects of bullying behaviors extend beyond the victims, in fact, the entire community is impacted. Consequences such as suicides and school shootings prompt researchers to push for a better knowledge of the dynamics, but victims face many more issues before these extremes. Victims routinely experience negative social and psychological effects such as constant anxiety, fear, and depression, even beyond the period of time in which the bullying occurred [3]. Their academic performance suffers as they become reluctant to go to school and face the bullies [2]. Significantly, research shows that they are also prone to becoming bullies themselves. The bullies, while powerful in their role, never learn the social skill of negotiation, instead solving their problems through manipulation and violence [3]. Adolescents and adults rely much more on negotiation skills, and bullies suffer because they lack these skills as they grow up. This leads to destructive behaviors that may result in consequences injury, addiction, and incarceration [3]. After a relatively easy childhood, bullies tend to struggle in adulthood.

The bystanders' academics are also disrupted by the instances of bullying but they also glean negative lessons from the bullying they witness, "Children who see bullying, day after day, absorb harmful lessons: bystanders should not intervene; victims deserve their fate; power beats fairness; adults do not care about children" [3]. In short, bullying affects the community, and its prevalence warrants the study of a mathematical model with the goal of generating new knowledge about interpersonal relations characterized by bullying and the consequences of bullying behaviors.

Bullying is a human dynamic that has been extensively studied by psychologists and sociologists. Typically, these studies use statistical methods to determine the prevalence of the behavior, but no mathematical models have been explored to study the dynamics of the bullying behavior in a population. The statistical modeling that has been done on this topic employs latent class analysis [10], and computer scientists have studied a cyberbullying scenario using an SIR model [14]. Neither of these studies include an analysis of the equilibria, numerical simulations, or a stability and bifurcation analysis. In this paper we present a model, that we will use to analyze parameters, run numerical simulations, and calculate the basic reproductive number. We will then extend the basic model to explore intervention methods.

2 Literature Review

2.1 Prevalence of Bullying in Schools

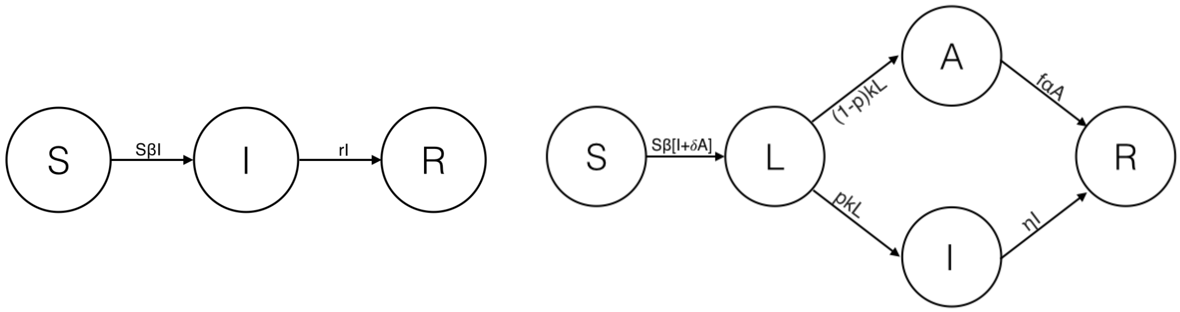
Most bullying research comes from psychology and sociology. These studies focus on the statistical prevalence of bullying behaviors, not on modeling the behaviors. Due to the complex and often subjective nature of bullying there is not a consensus on the typical amount of bullying that occurs in schools. In small specific studies it was found that the number of bullied children could be as low as 4% to as high as 82% [3]. More broadly, the percentage of children who are bullied around the world range from 10% [11] in a US study of over $15,000$ junior high students to 35% in a study of $3,000$ Lithuanian students and another of $6,000$ Maltan elementary and middle school students [6][3]. In the largest studies, with over $100,000$ participants, the range of values condenses to between 12% and 25% of children that reported being bullied [6][3].

The other aspect of the bullying dynamic is the prevalence of bullies. There is a greater level of consensus on the prevalence of bullies in schools across the world. At the extremes, in very specific studies the highest percentage of bullies in a population was 36% [6] compared to the extreme low of 3% [3]. In the larger studies the range narrows to between 8% [3] and 13% [11]. There are also small populations of those students who both are bullied and who bully others. This population is smaller than 10% [11] and typically even smaller than 5% [3]. Overall, psychology and sociology studies on bullying and childhood aggression report about 30% involvement in bullying behaviors, either as a bully, victim, or both [3].

These studies then extend to study the differing levels of bullying based on gender. Boys are more likely to engage in bullying behaviors than girls, by a factor of $5\% - 10\%$ [6]. Researchers have also studied how exactly bullying occurs: Physically, verbally, via social exclusion, by spreading rumors, or in the form of cyberbullying. "The prevalence rates of involvement in the five types of victimization were 13.2% for physical (male: 17.8%; female: 8.8%), 36.9% for verbal (male: 38.5%; female: 35.5%), 25.8% for social exclusion (male: 24.0%; female: 27.6%), 32.1% for rumor spreading (male: 27.6%; female: 36.3%), and 10.1% for cyber forms (male: 9.9%; female: 10.4%)" [16]. Cyberbullying is a new form of bullying, as it is unique to the technological age, and the ease of access children and adolescents have to the resources that are used to abuse other children. There is much less accountability to the tormentor. They don't have to justify their actions to a watching group of peers, nor are they necessarily recognized as the person doing the bullying. These types of bullies do not need to be as confident, and in fact it has been found that cyberbullies can out number cybervictims as much as three to one [3].

It is on this topic of cyberbullying that we found a single paper that focused on the modeling of bullying dynamic. The authors conceptualize a model that presents cyberbullying as a disease. They use a variation of a compartmental model, called a coflow diagram, to show, over time, how individuals move through the system. Their concept builds on the basic SIR model (Figure 1 (a)), where individuals begin as susceptible individuals (S) and after being infected, move to the infected class (I), then rehabilitate to move into the recovered class (R). Their conceptual models incorporate not only the victims, but the bullies and their recovery, the option for reinfection, and distinguishing between healthy and unhealthy contacts between bullies and victims. At no point do they clarify model parameters or write down a system of equations that match the coflows that they present. There is no mathematical analysis of the model [14].

We propose an epidemic model of ordinary differential equations to analyze bullying behavior. We deviate from the typical SIR model (Figure 1 (a)), instead taking inspiration from a SLAIR model used to analyze the spread of avian influenza (See Figure 1 (b)) [1].



(a) SIR Model for infectious diseases

(b) SLIAR Influenza Model

Figure 1: General epidemiological models for infectious diseases. Most models are derived from the SIR model. Our model will be a derivation of the SLIAR influenza model.

The SLIAR influenza model assumes two infectious states, one symptomatic state and one asymptomatic state. New infections are generated by interactions between susceptibles, S, and the symptomatic, I, and asymptomatic individuals, A. It also introduces a latent stage, L, of the infection before the individual becomes either symptomatic or asymptomatic individuals can then recover, R from either stage. These two elements reflect key aspects of the bullying dynamic. The primary advantage of this model over a typical SIR model is the three different states of infection. With this type of model it is possible to classify bullying and being bullied as infection states, matching the results of research on bullying and childhood aggression. Both groups of individuals are socially, emotionally, and/or physically damaged as a result of bullying behavior in their communities.

2.2 Current Models of Intervention

In Ken Rigby's book, *Bullying Interventions in Schools*, he discusses six basic approaches for stemming the tide of bullying in schools [12]. The most common approach for intervening in a population that is affected by bullying is the *Traditional Disciplinary Approach*. This tactic focuses solely on disciplining the perpetrators by making demands on the bullies that are backed-up by measures to ensure compliance. These measures aim to change behavior via fear and other negative motivators. This approach is popular in most schools due to the fear that most people have of bullies and their actions as well as its simplicity and longevity.

In contrast, another method that is used less frequently is *Strengthening the Victim* [12]. While it seeks to empower the victim, it does not have any procedures for disciplining the bully. This method aims to teach victims and potential victims skills or strategies to counteract the bullying behavior. In order to build confidence, the victims are taught physical and verbal skills, such as humor, witty comebacks and feigning disinterest. Many people perceive this as unfair. Designed to improve the victim's self esteem, there is a distinct lack of blame or accountability put on those that are misbehaving.

Four additional strategies involve mediation between the bully and the victim. The simplest is a simple *Mediation* [12], with impartial mediators, who may be the student's peers or an adult. The goal is to bring about a peaceful settlement or compromise, and the process is driven primarily by the students involved. It becomes very difficult to reach a settlement that both students are happy with and that does not validate bullying behavior, because the bully is often content with the situation as is, and they may try to force the victim to capitulate. However, if mediators are trained appropriately, basic mediation can be very successful.

An intervention strategy that addresses this distinct lack of accountability for the bully is the *Restorative Justice* intervention [12]. It is more direct: The method expects a behavioral change in the bully, and employs a range of procedures to ensure a just outcome. This is a future oriented approach and he desired outcome is less about retribution and more about healing hurt and putting the situation behind the community. The main criticism of this method is that it depends too heavily on the power of shame in transforming behavior. Psychologists have observed that exposing a child to shame has lasting detrimental effects. The success of this method hinges on the idea that the wrongdoer will want to restore their relationship with the whole community, but often the bully might already have the acceptance of the majority of their peers due to popularity or power.

The *Support Group Method* [12] has a similar goal as the *Restorative Justice Method*, but it does not assume that the feeling of shame is necessary to bring about a change in the wrongdoer. The method emphasizes the development of empathy so that the bully will be motivated to transform positively. Peers serving as mediators generate an environment in which the victim can safely convey their needs and feelings. They act as a group to provide backing for the victim, which pressures the bully to behave differently when interacting with the victim in the future. It requires a vast amount of preparation and support by the peers and adults, support that typically does not exist. This method was initially called the *No Blame Approach*, a name which generated significant backlash, resulting in a lack of support for the method by parents, teachers, administrators, and communities [12].

The *Method of Shared Concern* is the most complex and comprehensive method of addressing bullying which matches the complex nature of bullying [12]. It is a combination of basic mediation and the support group method. It adopts the no blame approach from the support group method and takes from mediation the idea that children in conflict must mediate to reach a peaceful settlement. Additionally, there is a crucial point of interaction between the mediator and the bully to ensure that the bully is prepared and willing to reach a mutually acceptable settlement. The mediation first begins with the practitioner interacting with the bully one on one to convey the plight of the victim without blame, attempting to motivate the bully to think about approaches the whole community could adopt to help the victim. After meeting with all the perpetrators, the adult then meets with the target to make it clear that actions have been taken to improve the situation. The group stage then progresses to first motivate the group to take on the solution as a community and then invite the victim to join the conversation about resolution. At this point the bully usually has experienced feelings of remorse or concern on behalf of the victim, and the process moves forward. Occasionally one side still harbors resentment towards the other and concessions must be made to reach a peaceful settlement. This method recognizes that all cases of bullying are not the same, necessitating a potentially time consuming no-blame process.

3 Model derivation

We propose a model for a population of children affected by bullying. The population is compartmentalized into five distinct subpopulations; Susceptible individuals, exposed individuals, bullies, non-bullies, and recovered individuals. Let $S(t)$ represent the number of individuals susceptible to bullying, $E(t)$ represent the number of individuals exposed to bullying, $B(t)$ represent the number of bullies, $N(t)$ represent the number of non-bullies, and $R(t)$ represent the number of individuals who have recovered from any influence of bullying at time t. Then the base model of the system is represented in Figure 2.

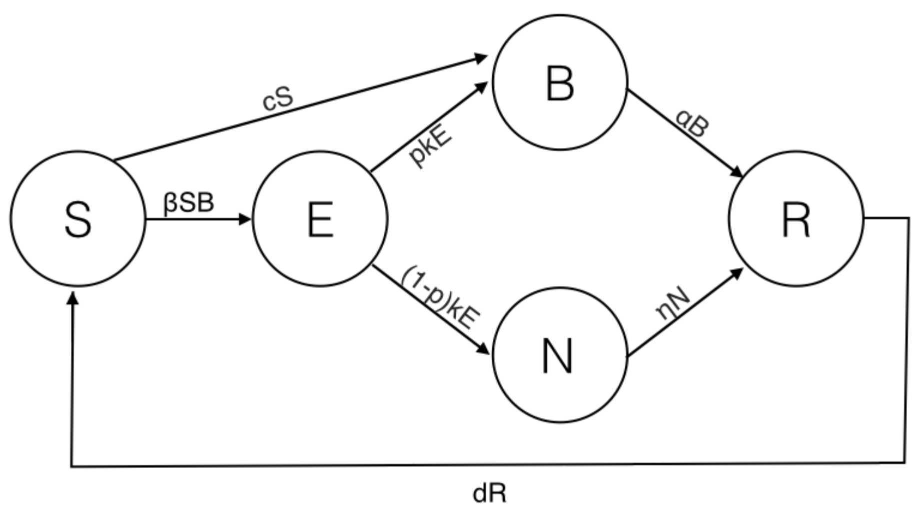


Figure 2: Our SEBNR Bullying Model that models relationships between, susceptible individuals (S), exposed individuals (E), non-bullies (N), bullies (B), and recovered individuals (R) in a closed population.

This leads to the following system of of equations:

$$\begin{aligned}
P(t) &= S + E + B + N + R \\
\frac{dS}{dt} &= -\beta SB + dR - cS \\
\frac{dE}{dt} &= \beta SB - kE \\
\frac{dB}{dt} &= pkE + cS - \alpha B \\
\frac{dN}{dt} &= (1-p)kE - \eta N \\
\frac{dR}{dt} &= \eta N + \alpha B - dR,
\end{aligned} \tag{1}$$

where β is the rate of "infection" via contact with a bully, k is the likelihood of remaining in the exposed state, p is the probability of becoming a bully after being bullied, α is the rate of recovery of bullies, η is the recovery rate of the non-bullies, d is the likelihood of remaining in the recovered state, and c is the probability of becoming a bully without first being exposed to bullying due to some other source of strain.

Table 1: Parameters and their Units

Parameter	Meaning	Units
β	rate of infection	$\frac{1}{\#ofindividuals \times time}$
c	proportion of susceptible population that spontaneously become bullies	$\frac{1}{time}$
p	probability of becoming a bully after being bullied	no units
k	$\frac{1}{k}$ is the amount of time spent exposed to bullying	$\frac{1}{time}$
η	$\frac{1}{\eta}$ is the amount of time spent as a non-bully	$\frac{1}{time}$
α	$\frac{1}{\alpha}$ is the amount of time spent as a bully	$\frac{1}{time}$
d	proportion of children that lose their immunity to bullying	$\frac{1}{time}$

Bullying is a complicated social dynamic, necessitating simplifying assumptions to be made in the course of designing our model. We assume that our population is sufficiently large and well mixed, and that there is no difference in how bullying impacts different ages, genders, socioeconomic groups, or ethnicities. The source of new bullies comes predominantly from those individuals who have been bullied previously due to a likelihood that an individual that has been bullied will become a bully themselves. This does not take into account other factors that contribute to the likelihood that an individual will become a bully. While this model does allow for a individual to spontaneously become a bully due to another source of strain, we assume that this is a small portion of the population [8].There are three subpopulations that are characterized as the diseased population, $E(t)$, $B(t)$, and $N(t)$, however we assume that the bullying population, $B(t)$, is the only population that is able to pass on and spread the "disease". We assume that the spread of bullying is best described through a mass action infection term. Additionally we assume that both bullies and non-bullies recover. The bullies recover from the bullying behavior and the non-bullies from the negative repercussions of being bullied. Recovered bullies and non-bullies are in the same compartment of recovered individuals, but they recover at different rates, α and η respectively. The assumption that immunity does not last for the recovered class of individuals has a large impact on the dynamics of the system, but is necessary due to the social dynamics that govern the population.

4 Model analysis

4.1 Basic dynamic properties

Now that our system of differential equations have been defined, it must be confirmed that they are consistent mathematically with what the equations describe. We will do this by making sure that the system is forward invariant, bounded, and that solutions exist and are unique.

4.1.1 Forward Invariance

Forward invariance is important to a system that describes a real-world population because the system needs to maintain non-negativity. It must be verified that if there are no individuals in a class that no individuals can leave that class and cause the population to become negative. Furthermore if a class of individuals starts off with a positive number of individuals it can never be negative.

Theorem 4.1. *Let $\mathbb{R}^5_+ = [0,\infty)^5$ be the cone of non-negative vectors in $\mathbb{R}^5$. Let $S(0)$, $E(0)$, $B(0)$, $N(0)$, and $R(0) \in \mathbb{R}^5_+$ be the initial conditions to system (1). Then the subspace of all positive initial conditions for (1) is forward invariant.*

Proof. The system of differential equations (1) can be written in the form

$$x' = F(t,x)$$

with $x = (S, E, B, N, R)$ and $F(t,x) = (S'(t), E'(t), B'(t), N'(t), R'(t))$.

It can be confirmed that $F_j(t,x) \geq 0$ whenever $t \geq 0$, $x \in \mathbb{R}^5_+$, and $x_j = 0$ for all $j = 1, \dots, 5$ due to the assumption that all parameters are written positively:

If $x_1 = S = 0$ and $x \in [0,\infty)^5$ then $F_1(t,x) = dR$ which implies that $F_1(t,x) \geq 0$.

If $x_2 = E = 0$ and $x \in [0,\infty)^5$ then $F_2(t,x) = \beta BS$ which implies that $F_2(t,x) \geq 0$.

If $x_3 = B = 0$ and $x \in [0,\infty)^5$ then $F_3(t,x) = pkE + cS$ which implies that $F_3(t,x) \geq 0$.

If $x_4 = N = 0$ and $x \in [0,\infty)^5$ then $F_4(t,x) = (1-p)kE$ which implies that $F_4(t,x) \geq 0$ since $p < 1$.

If $x_5 = R = 0$ and $x \in [0,\infty)^5$ then $F_5(t,x) = \eta N + \alpha B$ which implies that $F_5(t,x) \geq 0$.

Since each F_j is differentiable, each F_j is continuous. Thus if F_j is decreasing from a positive value, it must assume a value of zero before becoming negative. As a result, if starting at a positive initial condition x_j cannot become negative since for all $j = 1, \dots, 5$, $F_j(t,x) \geq 0$ when $t \geq 0$, $x \in \mathbb{R}^5_+$, and $x_j = 0$. Therefore the subspace of all positive initial conditions to (1) is forward invariant. □

4.1.2 Boundedness

Boundedness is essential because a real-world population cannot grow without bound. In our system it makes sense that the population is bounded because over a period of time a school has a fixed number of students. We will confirm that the population described by system (1) is bounded by the initial population of individuals in the system.

Theorem 4.2. *The solutions to the system (1) are bounded below by zero and bounded above by $P(0)$.*

Proof. By Theorem 4.1 the subspace of positive initial conditions is forward invariant, thus the solutions are bounded below by zero.

Next, by adding the right hand sides of the equations in system (1) we see that:

$$\frac{d}{dt}P = S'(t) + E'(t) + B'(t) + N'(t) + R'(t) = 0.$$

This implies that $P(t) = P(0)$ for all t. As S, E, B, N, and R are non-negative and add up to P, they are bounded by $P(0)$. Thus the solutions to (1) are bounded above by $P(0)$. □

4.1.3 Existence and Uniqueness of Solutions

The system of equations (1) is bounded and forward invariant. Now we must confirm that solutions to the system exist and that for each parameter set those solutions are unique.

Theorem 4.3. *For any S_0, E_0, B_0, N_0, and $R_0 \geq 0$, there exists a unique solution S, E, B, N, and R respectively, defined on $[0,\infty)$ to (1) satisfying $S(0) = S_0$, $E(0) = E_0$, $B(0) = B_0$, $N(0) = N_0$, $R(0) = R_0$ and $S(t)$, $E(t)$, $B(t)$, $N(t)$, and $R(t) \geq 0$.*

Proof. It must be shown that $F_j(t,x) \geq 0$ whenever $t \geq 0$, $x \in \mathbb{R}^5_+$, and $x_j = 0$, and that $F(t,x)$ is locally Lipschitz.

First, by Theorem 4.1 the subspace of positive initial conditions for (1) is forward invariant and so (1) satisfies the condition that for all $j = 1, \ldots, 5$, $F_j(t,x) \geq 0$ whenever $t \geq 0$, $x \in \mathbb{R}^n_+$, and $x_j = 0$.

Let x be a solution to (1) on some time interval $[a,b] \in \mathbb{R}_+$. Next, since $F(t,x)$ is a vector-valued function made up of combinations of functions that are continuous on the interval $[a,b]$ and differentiable on the interval (a,b) $F(t,x)$ itself is continuous and differentiable on those intervals. Therefore, by the Mean Value Theorem, there exists some $c \in [a,b]$ such that

$$\frac{|F_j(b) - F_j(a)|}{|b-a|} < F_j'(c) = M.$$

Thus for any interval (a,b), each $F_j(t,x)$ is locally Lipschitz. Since all of its components are locally Lipschitz, $F(t,x)$ is locally Lipschitz. Therefore the existence and uniqueness for solutions of (1) follows from Theorem A.1 (see Appendix).

$\square$

4.2 Equilibria

System (1) has two possible equilibrium that the sytem can reach if the system starts with an initial population $S(0) = \bar{S}$. The trivial equilibrium is found if $\bar{S} = 0$,

$$E_0 = (0,0,0,0,0).$$

The endemic equilibria is

$$E = \left(\bar{S}, \frac{c\beta\bar{S}^2}{k(\alpha - \beta p\bar{S})}, \frac{c\bar{S}}{p\beta\bar{S} - \alpha}, \frac{c\beta(1-p)\bar{S}^2}{\eta(p\beta\bar{S} - \alpha)}, \frac{c\bar{S}(p\beta\bar{S} - \alpha - \beta\bar{S})}{d(\alpha - p\beta\bar{S})}\right).$$

This equilibrium has many different interpretations if certain parameters are set to zero. First if $c = 0$, implying that the only source of new bullies comes from individuals that have been bullied themselves, then the equilibrium becomes:

$$E = (\bar{S}, 0, 0, 0, 0).$$

Next if $d = 0$, implying long term immunity to bullying then we find the equilibrium in which every person recovers:

$$E_1 = (0,0,0,0,\bar{R}).$$

We attempted to analyze the stability of each of the equilibria that we found using the Jacobian, however this analysis did not yield any insight into the stability of the equilibrium as the eigenvalues of the Jacobian at each point were 1.

4.3 Parameter Analysis

System (1) generates several different behaviors depending on parameter combinations. This allows the system to reach different equilibrium, representing a different breakdown of the population into the subpopulations for each parameter combination. Figure 3 shows a typical numerical simulation with an initial population of 100 individuals. This population has reached an endemic equilibrium in which bullying persists and constantly impacts the population. In this numerical simulation the system reaches an equilibrium where 68% of the population is made up of recovered individuals, 15% susceptible individuals, 8% non-bullies, 7% exposed individuals and 2% bullies. The exposed and non-bullying populations reach a maximum of 30%, while the bullying population surges to 5% after five days.

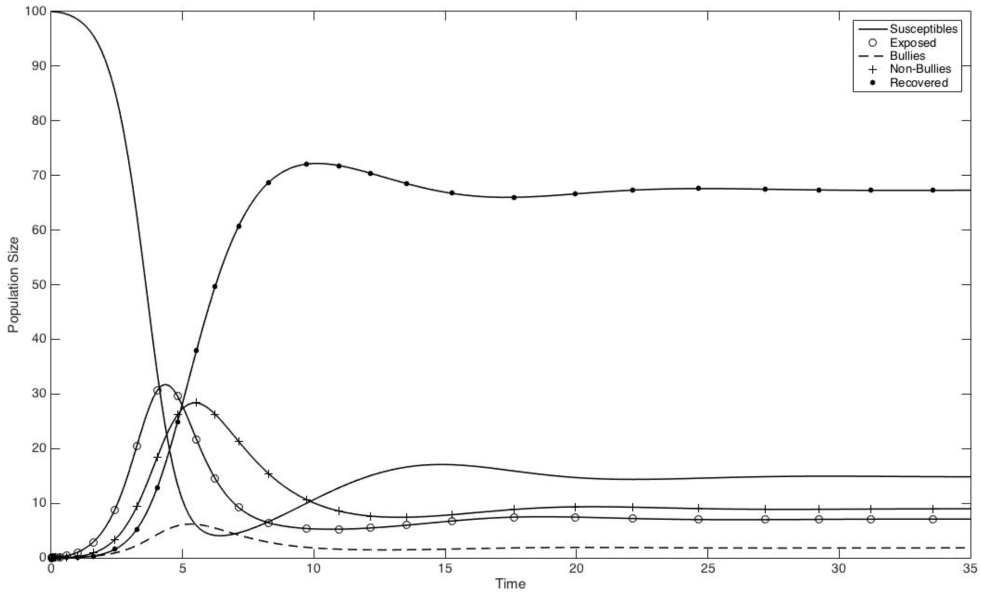


Figure 3: Numerical simulation of the model for an initial population of 100. Parameter Values are $\alpha = .75$, $\beta = .25$, $p = .2$, $c = .001$, $k = .95$, $\eta = .6$, and $d = .1$. This is an example of a system that reaches an equilibrium at which bullying persists.

Figure 4 shows that changes in the infection rate, β, significantly impact the dynamics of the system below a certain threshold. At this parameter combination, this threshold is at approximately $\beta = .2$. Below this threshold a small decrease in the infection rate has a noticeable effect on the equilibrium populations, lowering the exposed, non-bullies, and recovered subpopulations, while increasing the susceptible subpopulation at equilibrium. The bullying subpopulation is less impacted due to the small population size at any value of β, but for $\beta < .2$ a small decrease in β results in a slight decrease in the size of the bullying subpopulation at equilibrium. In contrast to systems with $\beta < .2$, large changes in the infection rate for a system with a $\beta > .2$, result in trivial changes to the equilibrium populations.

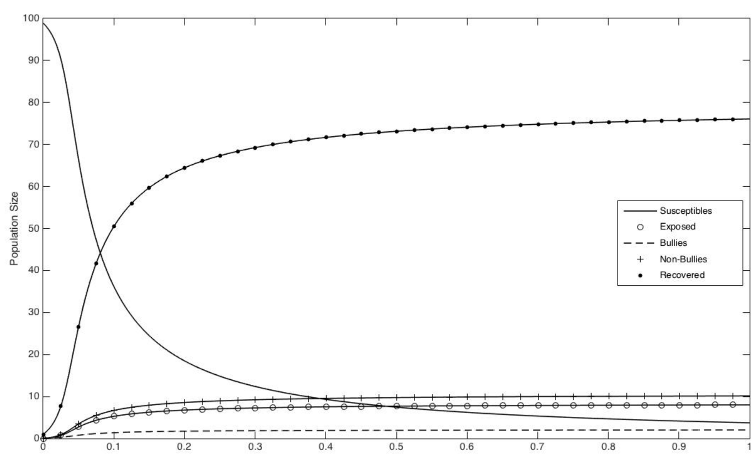


Figure 4: Equilibrium population sizes determined by changes in β for an initial population of 100. Common parameters are $\alpha = .75$, $p = .2$, $c = .001$, $k = 95$, $\eta = .6$, and $d = .1$. For systems with $\beta < .2$ the susceptible, exposed, non-bullying, and recovered subpopulations are sensitive to changes in β, whereas systems with $\beta > .2$ are not sensitive. The bullying subpopulation is less impacted due to the small population size at any value of β.

Figure 5 shows that changes in recovery rate of bullies, α, also impact the dynamics of the system at equilibrium below a certain threshold. At this parameter combination, this threshold is at approximately $\alpha = .1$. Below this threshold a small decrease in the infection rate has a noticeable effect on the equilibrium populations, lowering the exposed, non-bullies, and recovered subpopulations, while increasing the bully subpopulation at equilibrium. In contrast, large changes in the infection rate for a system with a $\beta > .2$, result in trivial changes to the equilibrium populations, except in two of the subpopulations. As α increases the size of the recovered population at equilibrium begins to decrease after reaching its maximum for $.1 < \alpha < .2$. The susceptible population increases at a constant rate as α increases.

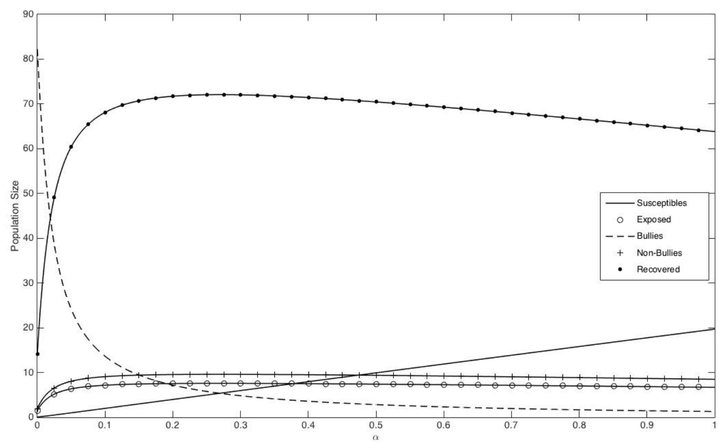


Figure 5: Equilibrium population sizes determined by changes in α for an initial population of 100. Common parameters are $\beta = .25$, $p = .2$, $c = .001$, $k = .95$, $\eta = .6$, and $d = .1$. For systems with $\alpha < .1$, the population sizes of exposed, non-bullies, bullies and recovered individuals at equilibrium are sensitive to changes in α, whereas systems with $\alpha > .1$ are not sensitive. Changes in α have a consistent effect on the susceptible subpoulation.

The recovery rate of non-bullies, η, impact the size of the populations of exposed, recovered, bullies, and non-bullies at equilibrium. It does not have a noticeable effect on the size of the susceptible population at equilibrium (see Figure 6). The effect on the recovered and the non-bullies is significant for systems with $\eta < .3$. A small decrease in η results in a large increase of the non-bullying population and a large decrease of the recovered population at equilibrium. The effect on the exposed and the bullies is moderate for systems with $\eta < .1$. A small decrease in η results in a moderate decrease of the exposed and bullying population at equilibrium. For systems with $\eta > .3$ changes in η have trivial impact on the system.

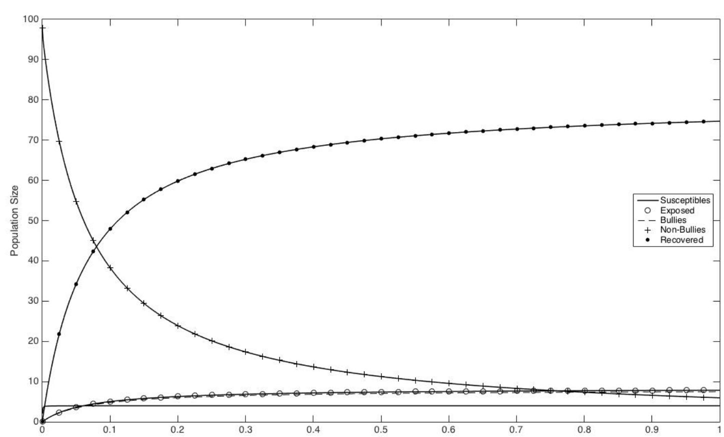


Figure 6: Equilibrium population sizes determined by changes in η for an initial population of 100. Common parameters are $\alpha = .75$, $\beta = .25$, $p = .2$, $c = .001$, $k = .95$, and $d = .1$. For systems with $\eta < .3$, the population sizes of non-bullies and recovered individuals at equilibrium are sensitive to changes in η. For systems with $\eta < .1$ population sizes of the exposed and bullying subpopulations are moderately sensitive to changes in η. For systems with $\eta > .3$ all population sizes at equilibrium are not sensitive to changes in η.

Changes in the probability of becoming a bully, p, have impact on the sizes of the bullying, non-bullying, and recovered subpopulations at equilibrium through the whole range of values, $0 < p < 1$, as can be seen in Figure 7. The relationship between the changes in p and the changes in the size of the bullying subpopulation is linear, where an increase in p corresponds to a proportional to the change in the number of bullies. The recovered and susceptible populations are very sensitive to changes when $p < .15$. A small decrease in p results in a large decrease in the number of recovered, and a large increase in the number of susceptible individuals. Additionally, as p begins to increase the number of recovered individuals will reach a maximum and then adopt a linear relationship with p, where an increase in p is proportional to the subsequent decrease in the number of recovered individuals. Systems with $p < .1$ are are moderately sensitive to how p changes, with respect to the exposed and non-bullying subpopulations. A small change in p generates a moderate change in the number of exposed individuals and non-bullies. Similar to the relationship that the recovered subpopulation and p display after the population reaches its maximum, the non-bullying subpopulation also becomes linearly related to p.

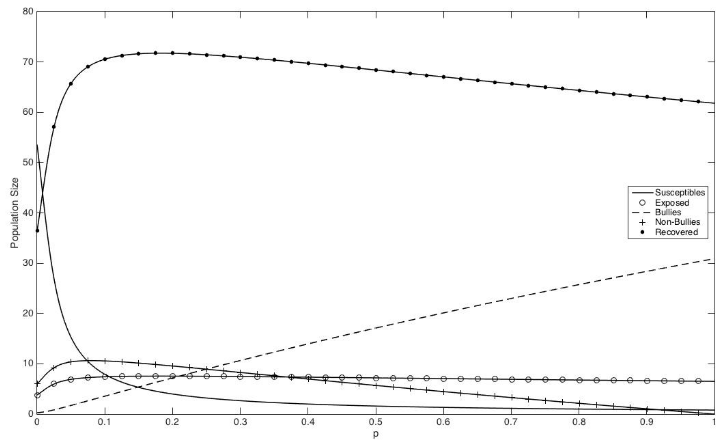


Figure 7: Equilibrium population sizes determined by changes in p for an initial population of 100. Common parameters are $\alpha = .75$, $\beta = .25$, $c = .001$, $k = .95$, $\eta = .6$, and $d = .1$. For systems with $p < .15$, the population sizes of susceptibles, exposed, non-bullies and recovered individuals at equilibrium are sensitive to changes in p. For systems with $p > .15$ the populations sizes of exposed and susceptibles are not sensitive to changes in p. Changes in p have a consistent effect on the bullying subpoulation and on the recovered and non-bullying populations for systems with $p > .15$.

Figure 8 shows that the system's long term behavior is not significantly impacted by changes in c, the probability of spontaneously becoming a bully through other sources. This trend remains as the values of c range from values that are less than .1.

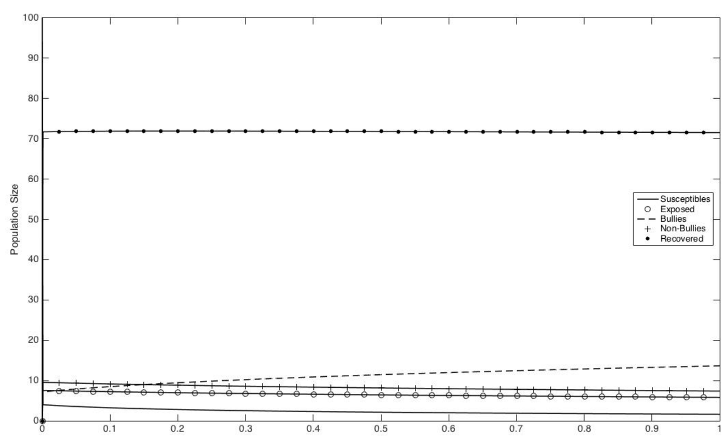


Figure 8: Equilibrium population sizes determined by changes in c for an initial population of 100. Common parameters are $\alpha = .75$, $\beta = .25$, $p = .2$, $k = .95$, $\eta = .6$, and $d = .1$. Systems with $0 < c < 1$ are not sensitive to changes in c.

In Figure 9 we let k vary from zero to one and observe the changes in equilibrium populations. If $k < .4$, small changes in k result in significant changes in the size of the exposed and recovered subpopulations at equilibrium. A small decrease in k results in an increase in the number of exposed individuals and a decrease in the number of recovered individuals. The number of non-bullies and bullies is moderately sensitive to changes to k for $k < .1$. A small decrease in k corresponds to a moderate decrease in both population sizes. The susceptible population is not sensitive to changes in k.

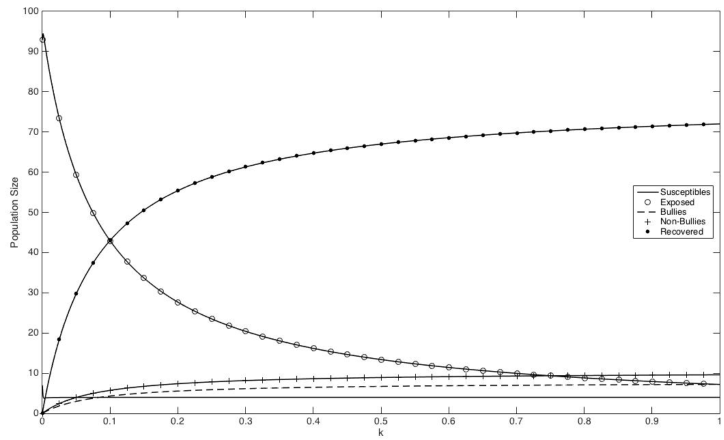


Figure 9: Equilibrium population sizes determined by changes in k for an initial population of 100. Common parameters are $\alpha = .75$, $\beta = .25$, $p = .2$, $c = .001$, $\eta = .6$, and $d = .1$. Systems with $k < .4$ show a sensitivity to changes in k for the sizes of the exposed and recovered subpopulations. Systems with $k < .1$ show a sensitivity to changes in k for the sizes of the non-bullying and bullying subpopulations. The size of the susceptible subpopulation is not sensitive to changes in k.

Changes in the proportion of recovered individuals that lose their immunity, d, have impact on the sizes of the exposed, bullying, non-bullying, and recovered subpopulations at equilibrium through the whole range of values, $0 < p < 1$, as can be seen in Figure 10. These populations are moderately sensitive to changes when $d > .35$ but this sensitivity increases when $d < .35$. In this range a slight decrease in d results in a larger decrease in the number of exposed individuals, bullies and non-bullies, while the number of recovered individuals increases. The susceptible population is not sensitive to changes in d.

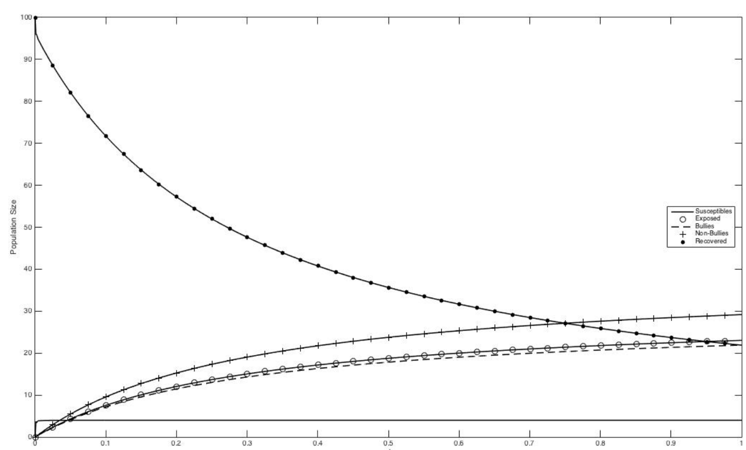


Figure 10: Equilibrium population sizes determined by changes in d for an initial population of 100. Common parameters are $\alpha = .75$, $\beta = .25$, $p = .2$, $c = .001$, $\eta = .6$, and $k = .95$. For systems with $0 < d < 1$ the sizes of the exposed, bullying, and non-bullying and recovered subpopulations are moderately sensitive to changes in d. This sensitivity increases when $d < .35$. The size of the susceptible subpopulation is not sensitive to changes in d.

From this analysis we can see that all the parameters besides c have substantial impact on the equilibrium population distribution, when they are changing on a range of low values. As a result the system is extremely sensitive to changes in parameter values of α, β, d, η, p, and k if the changes occur when the parameter is in the lower third of potential parameter values. If the parameter is changing in the range of values between .3 and 1 the system is much less sensitive, resulting in less change in the equilibrium distribution. The parameter d is an exception to this as the system remains sensitive to changes in d through out the range of possible values. Since the system is sensitive to most of the parameter values it will be valuable to explore which parameters describe the way in which bullying spreads through a population.

4.4 The Basic Reproductive Number: R_0

In infectious disease modeling we often want to find the parameter threshold at which the disease can and will persist in a situation in which a large population of susceptibles are exposed to a much smaller population of infected. This threshold depends on the average number of new infections that one infected individual causes in a population of only susceptibles. If this number is greater than one then the disease spreads, whereas if the typical number of new cases is less than one the disease will not spread. This average number of secondary infections is called R_0 or the basic reproductive number. In order to determine R_0 we must observe the system at the disease free equilibrium (DFE) which is found when $c = 0$ at $S = S_0$, $R = R_0$, and $E = B = N = 0$.

Following the method presented in [7] to calculate R_0 one must first understand the compartmental system in which it arises. In (1) there are five compartments, three of which contain infected individuals, and only one that is a *state-at-infection.* A *state-at-infection* is a state that an individual enters immediately after being infected. This group is crucial to understanding R_0 because this cohort of individuals will contain the newest group of infected individuals.

Adopting a deterministic model will not only simplify the model enough to apply methods from linear algebra, but it also matches the linear behavior close to the disease-free equilibrium. In this way we will look at successive generations of infected individuals. By observing the growth factor for each of these generations we find R_0. The growth potential of matrices are found via their eigenvalues. Thus we need to construct a matrix that properly describes our system at the DFE.

The first step is to determine the state(s) at infection and infected states. In this model there is only one state at infection E and three infected states E, B and N. These states will determine the infected subsystem:

$$\begin{aligned}\frac{dE}{dt} &= \beta BS - kE\\ \frac{dB}{dt} &= pkE - \alpha B\\ \frac{dN}{dt} &= (1-p)kE - \eta N.\end{aligned}$$

This system is not linear, so we linearize by setting $S = S_0$. The resulting linearized infected subsystem is:

$$\begin{aligned}\frac{dE}{dt} &= \beta S_0 B - kE\\ \frac{dB}{dt} &= pkE - \alpha B\\ \frac{dN}{dt} &= (1-p)kE - \eta N.\end{aligned}$$

If x is a vector describing the sizes of the infected populations E, B and N, then $\dot{x} = (T + \Sigma)x$. T is a 3×3 matrix describing new transmissions of the disease (those new individuals entering the state at infection E) and Σ is a 3×3 matrix representing the transitions in and out of the infected states. With this structure the entries of the matrix $-\Sigma^{-1}$ has the interpretation of how long individuals stay in each of the states depending on in which state they passed the last generation. The next generation matrix (NGM) K_L is defined as $K_L = -T\Sigma_{-1}$. This matrix is not the final NGM K because only the state(s) at infection have indication of the strength of the infection at the DFE. For the matrix T, element T_{ij} is the rate at which those individuals in state j give rise to new individuals in the infected state i. For this model,

$$T = \begin{pmatrix} 0 & \beta S_0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

T has one non-zero entry because only individuals in state 2 (B) give rise to individuals in state 1 (E). In order to define Σ we must remember that this is a subsystem of a larger system. Σ is related to matrices that determine Markov processes. Σ_{ij} for $i \neq j$ is the probability per unit of time for an individual's state to change from j to i. Without removal from the subsystem $\Sigma_{ii} = -\sum_{i \neq j} \Sigma_{ij}$. However if there is removal from the subsystem entirely then terms must be added to account for this. If in state i there is a removal rate γ from the subsystem as there is from B then $\Sigma_{ii} = -\gamma$. Thus for this model,

$$\Sigma = \begin{pmatrix} -k & 0 & 0 \\ pk & -\alpha & 0 \\ (1-p)k & 0 & -\eta \end{pmatrix} \quad \text{and} \quad \Sigma_{-1} = \begin{pmatrix} -\frac{1}{k} & 0 & 0 \\ \frac{p}{\alpha} & -\frac{1}{\alpha} & 0 \\ \frac{\alpha k - \alpha k p}{\alpha \eta k} & 0 & -\frac{1}{\eta} \end{pmatrix}.$$

Then using the logic from above,

$$K_L = \begin{pmatrix} -\frac{S_0 \beta p}{\alpha} & \frac{S_0 \beta}{\alpha} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To arrive at the actual NGM K one must use a matrix ε to isolate the aspects of K_L that refer to the state(s) at infection. In this model there is only 1 state at infection so $\varepsilon = (1,\ 0,\ 0)^T$. We then define the NGM $K = \varepsilon^T K_L \varepsilon$ and obtain,

$$K = \frac{S_0 \beta p}{\alpha}.$$

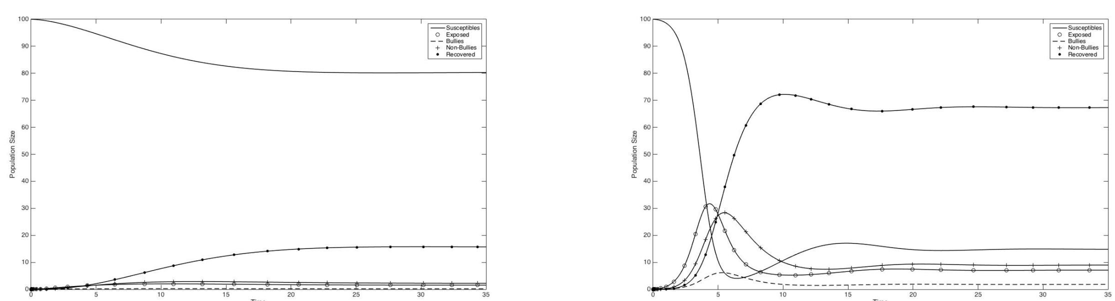
Since K is a 1x1 matrix, its only eigenvalue is its entry, thus for this model,

$$R_0 = \frac{S_0 \beta p}{\alpha}.$$

The initial population, S_0 is fixed so we can normalize the parameters to the population size, and simplify to get,

$$R_0 = \frac{\beta p}{\alpha}.$$

Figure 11 shows numerical simulations of systems with basic reproductive numbers less than one or greater than one. With parameters at unrealistic values we can generate a system that is at a DFE, however most parameter combinations result in an endemic equilibrium where bullying persists.



(a) Example of a system with $R_0 < 1$. $R_0 = 0.7645$ when the initial population is 100 and the parameter values are $\alpha = .99$, $\beta = .087$, $p = .087$, $c = 0$, $k = .95$, $\eta = .6$, and $d = .1$. This results in a disease free equilibrium.

(b) Example of a system with $R_0 > 1$. $R_0 = 6.667$ when the initial population is 100 and the parameter values are $\alpha = .75$, $\beta = .25$, $p = .2$, $c = .001$, $k = .95$, $\eta = .6$, and $d = .1$. This results in an endemic equilibrium.

Figure 11: Systems at different equilibrium states depending on the value of R_0.

We were able to run numerical simulations with a vast set of parameter combinations, and the norm was systems that reached an endemic equilibrium where bullying persists. This matches the intuition we have about R_0, because with a large enough population it is extremely difficult to offset that with low β and p values and a large α value, since all of the parameters in our model are between zero and one.

4.4.1 R_0 Sensitivity Analysis

We now would like to explore how sensitive R_0 is to changes in β, p, and α.

Definition 4.1. *The normalized forward sensitivity index of a variable, u, that depends differentiability on a parameter, p, is defined as [4]:*

$$\gamma_p^u = \frac{\partial u}{\partial p} \times \frac{p}{u}.$$

As R_0 depends on only three parameters, β, p, and α, the sensitivity analysis of R_0 to changes in the other parameters is zero. The sensitivity index for R_0 with respect to β, is

$$\gamma_\beta^{R_0} = \frac{\partial \beta}{\partial R_0} \times \frac{R_0}{\beta} = \frac{\partial \beta}{\partial R_0}\frac{\alpha}{S_0 p} = \frac{S_0 p}{\alpha}\frac{\alpha}{S_0 p} = 1.$$

The sensitivity index for R_0 with respect to p, is

$$\gamma_p^{R_0} = \frac{\partial p}{\partial R_0} \times \frac{R_0}{p} = \frac{\partial p}{\partial R_0}\frac{\alpha}{S_0 \beta} = \frac{S_0 \beta}{\alpha}\frac{\alpha}{S_0 \beta} = 1.$$

For S_0, β, and p a sensitivity index of 1 means that a 5% increase in S_0, β, or p will result in a 5% increase in R_0. The sensitivity index for R_0 with respect to α, is

$$\gamma_\alpha^{R_0} = \frac{\partial \alpha}{\partial R_0} \times \frac{R_0}{alpha} = \frac{\partial \alpha}{\partial R_0}\frac{\alpha^2}{S_0 \beta p} = \frac{S_0 \beta p}{-\alpha^2}\frac{\alpha^2}{S_0 \beta p} = -1.$$

The sensitivity index with respect to α is -1 meaning that a 5% increase in α will result in a 5% decrease in R_0. Through this sensitivity analysis, we can see that R_0 is equally sensitive to changes β and p while it is inversely sensitive to changes in α.

5 Intervention Model

5.1 Model Derivation and Assumptions

Now that the dynamics of the base model are better understood, next we explore the impact of interventions that schools can use to prevent and treat bullying. We first build a general model that can be tailored to match different interventions.

Let $S(t)$, $E(t)$, $B(t)$, $N(t)$, and $R(t)$, be defined as in the basic model. To incorporate intervention let $S_T(t)$, $E_T(t)$, $B_T(t)$, $N_T(t)$, and $R_T(t)$ be those individuals in each respective class that are undergoing an intervention. These individuals are still impacted by bullying behaviors but the effects are not nearly as severe. Then the model incorporating intervention is represented in Figure 12 is given by the system of ordinary differential equations in 2.

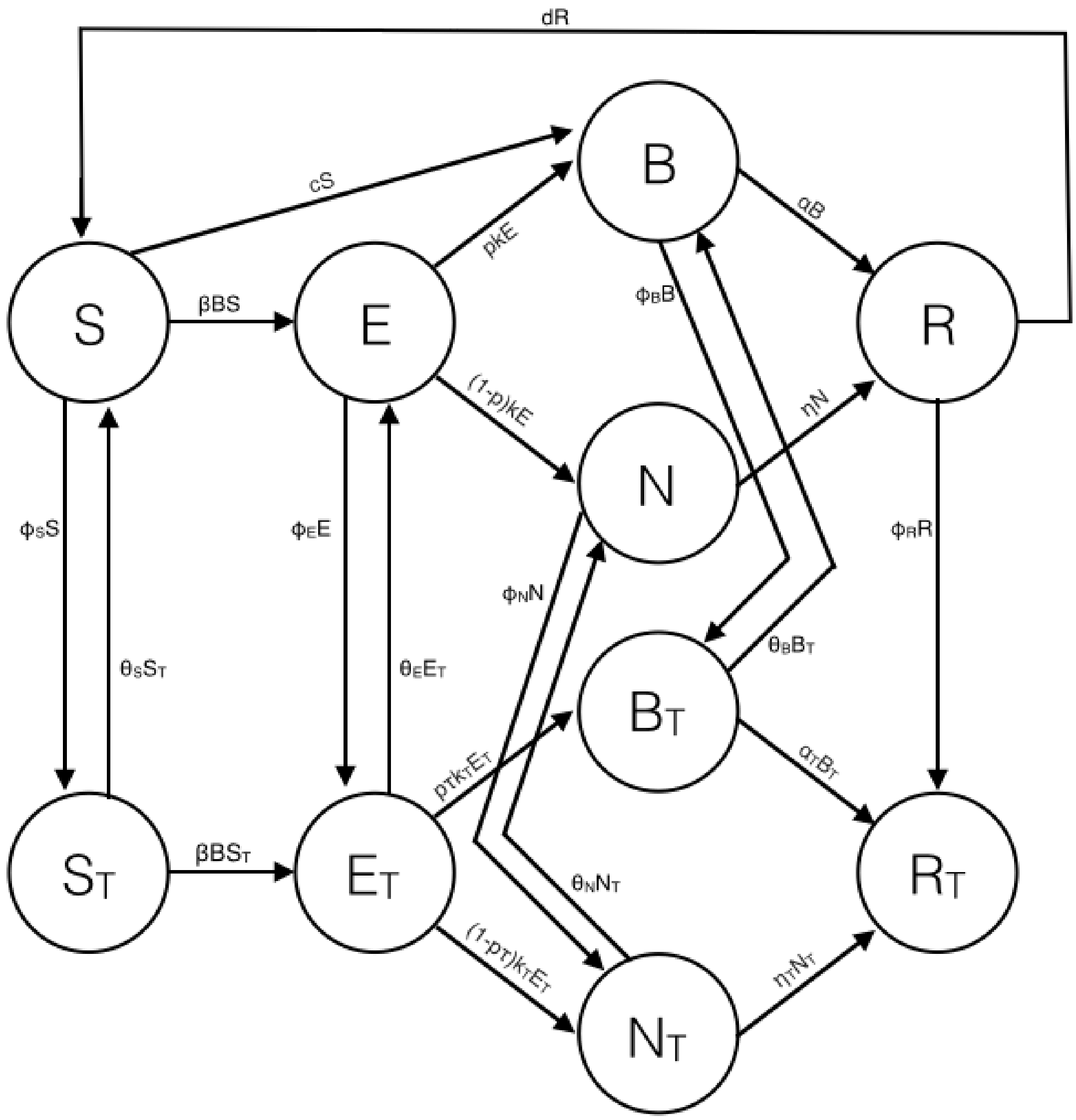


Figure 12: The general model incorporating intervention strategies that models the interactions of individuals that have not experienced intervention and those that have. This model can be tailored to match specific strategies used in schools today.

This leads to the following system of of equations:

$$
\begin{aligned}
P(t) &= S + S_t + E + E_t + B + B_t + N + N_t + R + R_t \\
\frac{dS}{dt} &= -\beta BS + dR - cS - \phi_S S + \theta_S S_T \\
\frac{dE}{dt} &= \beta BS - kE - \phi_E E + \theta_E E_T \\
\frac{dB}{dt} &= pkE + cS - \alpha B - \phi_B B + \theta_B B_T \\
\frac{dN}{dt} &= (1-p)kE - \eta N - \phi_N N + \theta_N N_T \\
\frac{dR}{dt} &= \eta N + \alpha B - dR - \phi_R R \\
\frac{dS_T}{dt} &= -\beta BS_T + \phi_S S - \theta_S S_T \\
\frac{dE_T}{dt} &= \beta BS_T - k_T E_T + \phi_E E - \theta_E E_T \\
\frac{dB_T}{dt} &= p\tau k_T E_T - \alpha_T B_T + \phi_B B - \theta_B B_T \\
\frac{dN_T}{dt} &= (1-p\tau)k_T E_T - \eta_T N_T + \phi_N N - \theta_N N_T \\
\frac{dR_T}{dt} &= \eta_T N_T + \alpha_T B_T + \phi_R R.
\end{aligned}
\tag{2}
$$

In this system the probability of intervention for each class is ϕ_s, ϕ_e, ϕ_b, ϕ_n, and ϕ_r respectively. Assuming that each intervention has a certain success rate, let θ_s,θ_e, θ_n, and θ_b be the probability that the intervention will fail. We assume that if an individual reaches the recovered class that has had a successful intervention, then the individual is permanently immune to the effects of bullying.

Following the assumption that the effects of bullying are less severe in each of the "treated" classes we reach two conditions on the system . First, the parameters that determine the rates of movement through the intervention classes are higher than the rates of the normal classes. Second, the probability of becoming a bully must be lower than that of those individuals that have not experienced intervention. Thus we will assume $\alpha_t > \alpha$, $k_t > k$, and $\eta_t > \eta$ while $\tau < 1$ will reduce the probability of becoming a bully by τ.

5.2 Parameter Analysis

Unlike like the model without intervention (1), in the model with intervention (2) there is only one equilibrium reached for non-zero parameter values. The equilibrium distribution of the population is entirely made up of the permanently "immune" recovered individuals. Instead of influencing equilibrium distributions the parameters in this model affect how quickly the population recovers fully and how badly the initial outbreak affects the population. There are 19 parameters in this system and as a result, if all parameter values are non-zero often if a parameter change results in a system behavior that overlaps with the system behavior that is generated by a different parameter change. Also, many of the intervention parameters cooperate and generate larger system changes if they are all slightly changed in tandem, than if only one is changed to a large degree. This overlap and cooperative behavior of the parameters makes a parameter analysis complicated and inconsistent. As a result, instead of a formal parameter analysis instead we will outline different behaviors that were observed as we ran numerical simulations over many different parameter combinations.

The presence of intervention changes the effects that the basic set of parameters has on the system. The infection rate β has very little effect on the speed of the recovery process once $\beta > .1$. The sole effect it has is on the maximum number of exposed individuals at the onset of the epidemic. Below that value the treatment process slows down significantly as less individuals are impacted by bullying. This results in more individuals transferring to the intervention side of the model from the susceptible class rather than the later classes. The recovery rate of bullies α has a small effect on the non-treated populations. At low values the susceptible population drops and the non-bullies, exposed and bully population rises rapidly, and then individuals recover via intervention from later stages, during or post being bullied. At high values, less of the susceptible population is affected at onset and more individuals undergo intervention pre-infection.

The lower the recovery rate, the faster the whole population recovers into the immune class. A low d, the proportion of children that loose their immunity, causes more of the population to recover from the recovered state as individuals accumulate in the untreated recovered class. A high d causes intervention and eventual immunity to occur more frequently from the pre-recovery states.

The probability of becoming a bully p affects the speed of the recovery of the entire population. While low enough p will allow the system to recover and not allow an epidemic to occur, the transition to a completely immune population is much slower. As p increases, the bullying epidemic will run through the population faster and more severely, affecting more people, but the entire population will become immune much faster as a result. The parameter d does not have a large effect on the speed at which the entire population becomes immune. Instead it determines whether individuals experience intervention while in the recovered state or if they undergo intervention in one of the other four classes. The probability of becoming a bullying without being bullied previously, c, as in the basic model does not impact the system on the range of values between zero and one.

The parameters η and k both have a very similar effect on the population under intervention as they affect the population in (1). The recovery rate of non-bullies, η has a large affect on the non-bully population and the recovered population. A low η results in more recovery via individuals in the non-bully stage undergoing intervention. High η results in the intervention occurring more often in the recovered stage, or if the individual becomes susceptible, form any of the other classes that it may pass through. η does not have a significant effect on the end dynamics, or the speed at which the population fully recovers. The parameter k determines whether an individual experiences intervention predominately while in the exposed state or otherwise. As individuals remain in the exposed state longer, reflected by a low k value, the populations of the other infected states drops dramatically. However as k increases these populations increase as well, while the exposed population will maintain a large maximum value but its long term population will be significantly lower.

The new parameters that determine the movement between the intervention classes and the non intervention parameters have similar effects on the system as their non-intervention counterparts. The recovery rate of the bullies that had experienced intervention, α_t, has a very similar affect as α, however, in the intervention model, as α_t approaches zero, it slows the speed at which the population reaches the immune class. It slows it to the point where it takes much longer than at high α_t(or even low α_t and high α). The rate of recovery of the treated non-bullies, η_t has a larger effect on the system if it cooperates with η. If either η_t and η are greater than .1 then changing η has little impact on the system, however when both are lower than .1 the full recover of the population into the immune class slows down as with α_t. A low k_t will motivate intervention to occur in a later stage than the exposed state whereas a high k_t just moves the recovery more and more quickly through the treatment stages. Low of both k and k_t slows the recovery of the population significantly, primarily due to the low k value and not the low k_t value. The parameter that reduces the probability of becoming a bully after being bullied, τ, has no visible effect on long or short term dynamics, unless p is also altered, the change in the system is the same as if only p was changed.

The parameters that determine the success of the interventions at each stage, θ_s,θ_e, θ_n, and θ_b, have nearly trivial effects on population due to the likelihood of treatment in a subsequent stage that would be successful. If low θ is combined with any number of other altered parameters that slow down intervention in any way, there is a small additional slow down of full recovery of the population. Collectively they have more effect if all the θ's are either high or low as there is a build up of impact. The parameters ϕ_s, ϕ_e, ϕ_b, ϕ_n, and ϕ_r which determine the rate at which individuals undergo intervention have the effect of speeding up or slowing down the rate of intervention, but collectively they have much more impact on the system if they are all high or all low. If all the ϕ's are high full recovery of the population is extremely fast and if the ϕ's are low then the population recovers slowly. The rate of intervention of the recovered class, ϕ_r, has the largest effect on the slowing or speeding up the recovery of the population. Even combined when each of the θ are high and each of the ϕ are low the population still recovers relatively quickly.

From our observations, most parameters have impact on the system, besides τ and c, however similiar system behaviors can be generated with different sets of parameter values. Additionally there are groups of parameters that impact the system by cooperating rather than affecting the system independently from the other parameters. To better understand the system we will now allow some of the ϕ and θ to be zero and explore at how these parameter combinations reflect interventions used in schools.

5.3 Implications of Differing Intervention Strategies

Worldwide, schools respond to bullying in different ways. We will study the impact of the six intervention strategies given in [12]. We will focus on these six interventions as they are direct responses to bullying behavior, rather than the general anti-bullying lessons taught at most schools as these affect the whole population by changing parameter values. A summary of the six intervention strategies can be found in Table 2.

Table 2: Bullying Intervention Strategies

Intervention Strategy	Description
Traditional Disciplinary Approach	Disciplinary action is focused on the bully in order to change the bullying behavior. Most common strategy due to simplicity, fairness, and longevity.
Strengthening the Victim	Strategies to teach victims or potential victims skills to deflect bullying directed at them. These can be physical skills, verbal and communication skills or improving self confidence.
Mediation	Moderated by impartial mediators, the victim and the bully work together to decide on a settlement that both parties are happy with. Hinges on the idea that neither party is happy with the current situation.
Restorative Justice	Another mediation strategy that expects a change in the bully's behavior in order to bring about a better future for the community. Shame can be a driving factor in the behavioral change.
Support Group Method	A mediation strategy that tries to build up the bully's ability to empathize with the victim, hoping empathy will motivate behavioral change. Formerly known as the No Blame Method.
Method Of Shared Concern	A multistage mediation strategy in which peers and adult moderators motivate the bully to empathize with the victim and come up with a solution that benefits the whole community. The victim is then invited to share input on the strategy.

In the next sections we will set up the model for each of the six interventions, clarifying assumptions that are made and choices for parameter values. In most of the model all the opportunities that an individual has to receive intervention are not used, because each method of intervention uses the schools resources to target different pathways of intervention. Eleven of the parameters from the intervention model will remain the same in each of the intervention strategies. These parameters can be seen in Table 3. Note that in every numerical simulation of the intervention model (2) the population will reach an equilibrium of a 100% permanently immune population, if any intervention is incorporated. The changes in the parameters determine how quickly it does so, and how impacted the population is at the onset of the bullying behaviors. The level to which the population has become permanently immune and how impacted the system is at the onset is how we will study the effectiveness of a particular intervention.

Table 3: Standard Parameters used in all numerical simulations.

Parameter	α	β	η	c	d	p	τ	k_t	η_t	α_t
Value	.75	.25	.6	.001	.1	.2	.9	.99	.66	.825

5.3.1 Traditional Disciplinary Approach

The Traditional Disciplinary Approach is the most commonly used and highly accepted form of intervention used today in schools. Due to its popularity and prevalence it makes the method easier to adopt as a "whole school" practice, the linchpin in a successful anti-bullying program [12][5]. This method focuses entirely on changing the behavior of the bullies by assigning the appropriate punishment to the bully. This is reflected in ((2)) by letting ϕ_s, ϕ_e, ϕ_n, and $\phi_r = 0$ to show that there is no possibility of intervention if an individual is in one of those four classes. Letting the respective failure rates stay the same has no effect on the model, as the only class they can fail from is the B_T class. Assuming that 50% of bullies undergo an intervention ($\phi_b = .5$) and 50% of the interventions are successful ($\theta_b = .5$), the resultant dynamics are shown in Figure 13. The model is run for 100 days and we can see that at that time 30% of the population is permanently immune to the effects of bullying. We also see that there is a large impact initially when the bullying behavior is introduced into the population. There is a peak of exposed and non-bullies that reaches 25% of the population after five days. These populations also remain at 5% of the population as they slowly decrease as the population tends towards permanent immunity.

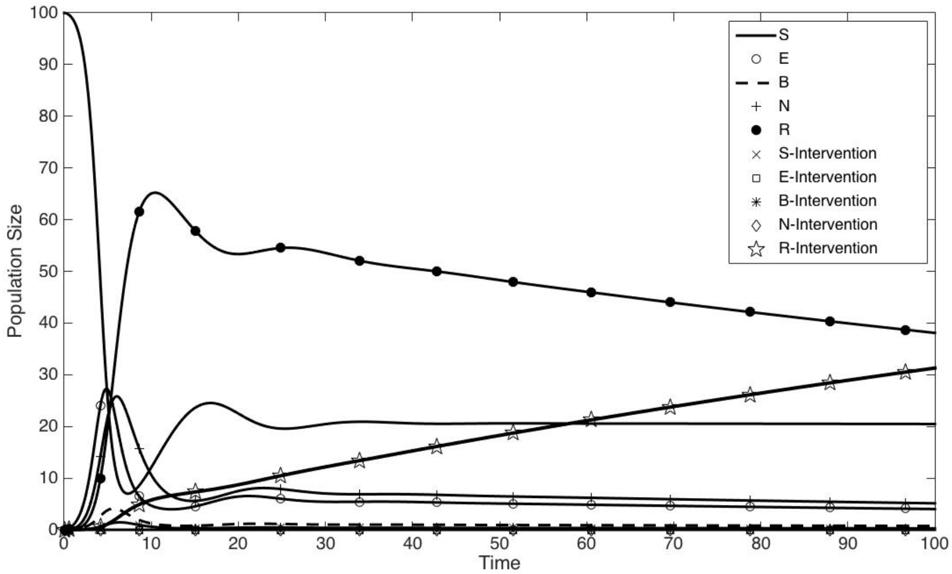


Figure 13: Numerical simulation of an implementation of the Traditional Disciplinary Approach that assumes that 50% of bullies undergo intervention and that 50% of those interventions fail. After one-hundred days the population is only 30% recovered.

5.3.2 Strengthening the Victim

In contrast to the Traditional Approach, the strategy of Strengthening the Victim ignores the effect of the bully on the recovery. Instead the intervention is geared to aid the victims or potential victims, by teaching them tools to interact with the bullies without being impacted to the same extent as the victims that have not gone through intervention. In the simple model (1), any the subgroups $S(t)$, $E(t)$, and $N(t)$ may be strengthening. This can be reflected in the model with intervention,((2)), in a few different ways. First, we assume that only the susceptible population is being strengthened. To achieve this in (2) let ϕ_e, ϕ_b, ϕ_n, and $\phi_r = 0$ to show that there is no possibility of intervention if an individual is in one of those four classes. We assume that 50% of the susceptible population undergo an intervention, then $\phi_s = .5$. We maintain the assumption that an individual's intervention can fail at any stage of the infection by setting θ_s, θ_e, θ_b, and $\theta_n = .5$. The resultant dynamics are shown in Figure 14. We see that after the same period of 100 days over 60% of the population has become permanently immune. We also see that the exposed and the non-bullying populations reach a peak of only about 15% and 20% respectively after five days, and they remain at 5% as the system tends towards immunity.

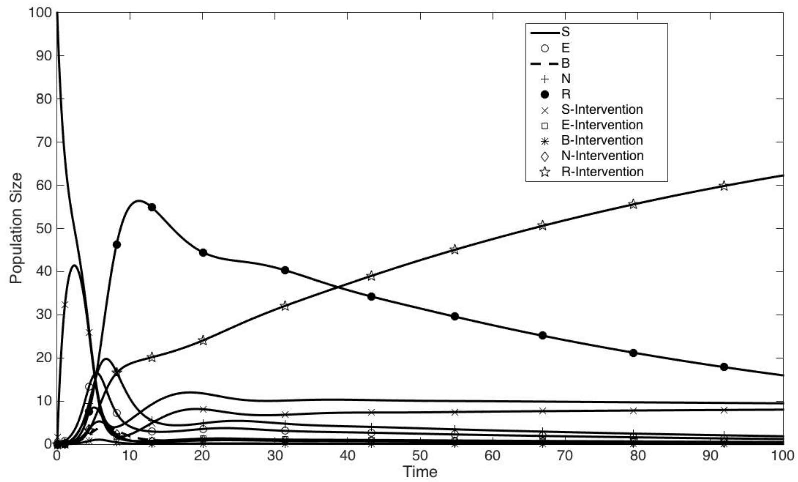


Figure 14: Numerical simulation of an implementation of the Strengthening the Victim that assumes that 50% of susceptible individuals undergo intervention and that 50% of those interventions fail. After fifty days the population is 60% recovered.

Next we assume that the susceptible and the exposed population are being strengthened. To achieve this in the model with intervention,(2), let ϕ_b, ϕ_n and $\phi_r = 0$ to show that there is no possibility of intervention if an individual is in one of those three classes. We assume that 50% of the victim population undergo an intervention, then $\phi_s = \phi_e = .5$. Setting θ_s, θ_e, θ_b, and $\theta_n = .5$ maintains the assumption that an individual's intervention can fail at any stage of the infection with a probability of .5. The resultant dynamics of the system are shown in Figure 15.We see that after the same period of 100 days over 70% of the population has become permanently immune. We also see that the exposed and the non-bullying populations reach a peak of only about 10% after five days, and they remain below 5% as the system tends towards immunity.

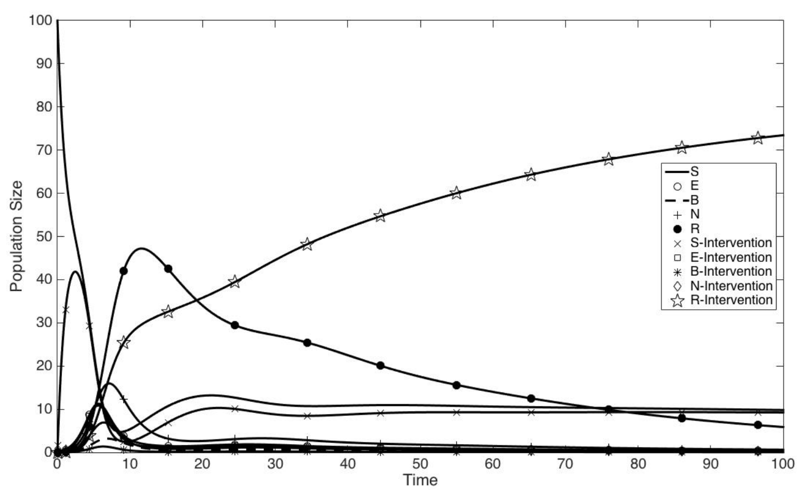


Figure 15: Numerical simulation of an implementation of the Strengthening the Victim that assumes that 50% of susceptible and exposed individuals undergo intervention and that 50% of those interventions fail. After fifty days the population is 70% recovered.

Finally we assume that the susceptible, the exposed, and the non-bullies population are being strengthened. To achieve this in the model with intervention, 2, let ϕ_b and $\phi_r = 0$ to show that there is no possibility of intervention if an individual is in one of those two classes. We assume again that 50% of the victim population undergo an intervention, then $\phi_s = \phi_e = \phi_n = .5$. Setting θ_s, θ_e, θ_b, and $\theta_n = .5$ maintains the assumption that an individual's intervention can fail at any stage of the infection with a probability of .5. The resulting dynamics of the system are shown in Figure 16.We see that after the period of 100 days over 80% of the population has become permanently immune. We also see that the exposed and the non-bullying populations reach a peak of only about10% after five days, and they remain at well below 5% as the system tends towards immunity. It also only takes 20 days for the community to reach 50% immunity, meaning that three weeks of focus on a bullying intervention could have a significant impact on a school.

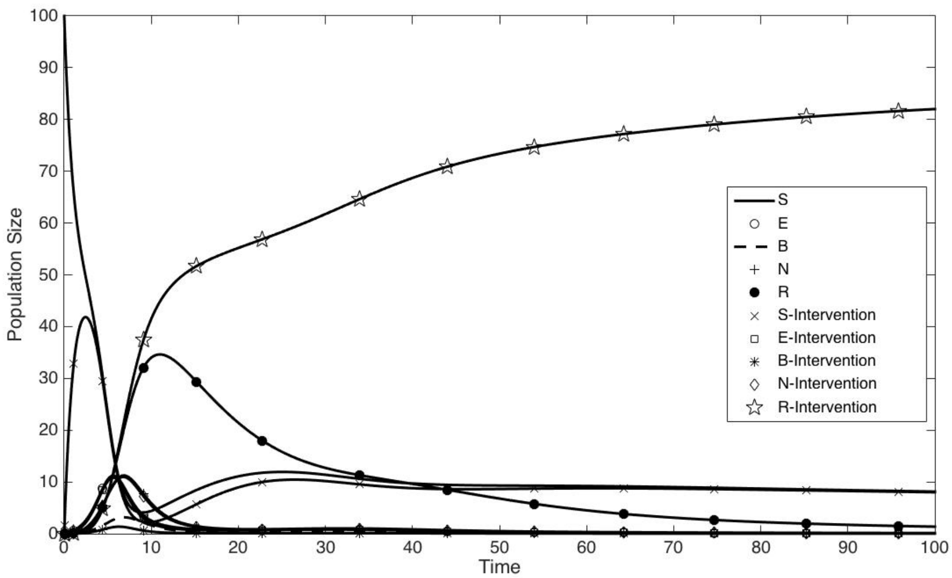


Figure 16: Numerical simulation of an implementation of the Strengthening the Victim that assumes that 50% of susceptible, exposed and non-bullying individuals undergo intervention and that 50% of those interventions fail. After fifty days the population is 80% recovered.

5.3.3 Mediation

The method of mediation is the first in the set of interventions that hinge on positive interactions between the bully and the victim. In order to reflect the interaction in (2) a mass action term is introduced. Instead of a constant proportion of the exposed and the bully population undergoing intervention and transferring classes, the rate of intervention depends on the effectiveness of an interaction. As such ϕ_s, ϕ_n, and $\phi_r = 0$ to show that there is no possibility of intervention if an individual is in one of those four classes. We maintain the assumption that an individual's intervention can fail at any stage of the infection by setting θ_s, θ_e, θ_b, and $\theta_n = .5$. Then let $\phi_e = m_1 B(t)$ and $\phi_b = m_2 E(t)$. Based on the researched effectiveness of the intervention, we set $m_1 = m_2 = .25$. These low values are due to the lack of accountability to motivate the bully to change and the lack of support for the victim that allows the intervention to often fail in comparison to the other mediation based methods [12]. The resulting dynamic is shown in Figure 17. The population is 50% immune after 100 days of intervention, and the maximum number of exposed and non-bullies is reached after five days and is just over 10%. One can notice that the level that the Susceptible individuals does not drop as drastically as in the previous models. This can be interpreted that more individuals are not being impacted by bullying behaviors and then recovering, instead they are undergoing intervention prior to experiencing bullying.

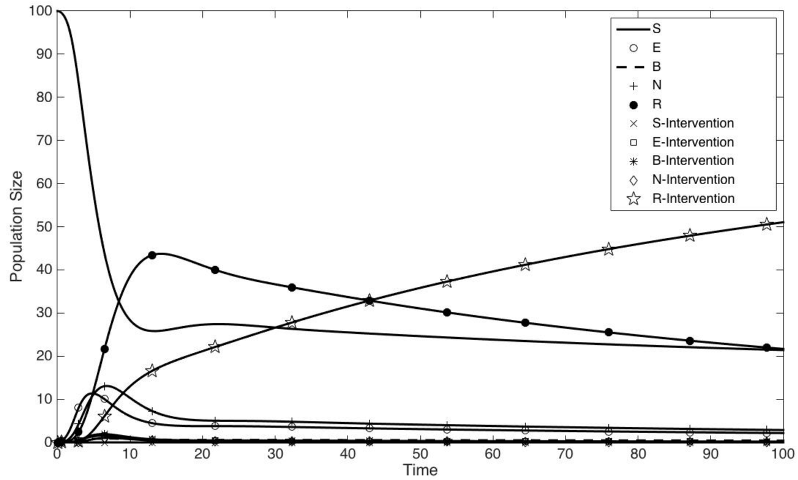


Figure 17: Numerical simulation of an implementation of the Mediation Strategy that assumes that 25% of the interactions between a bully and a victim are mediated and that 50% of those interventions will not result in a satisfactory compromise for each individuals. After fifty days the population is 50% recovered.

5.3.4 Restorative Justice

The method of Restorative Justice is the second in the set of interventions that hinge on positive interactions between the bully and the victim. This method focuses more on trying to influence the perpetrators to repent and change their behaviors rather than attempting to support the victim in advocating for change. Maintain ϕ_s, ϕ_n, and $\phi_r = 0$ and θ_s, θ_e, θ_b, and $\theta_n = .5$. Then let $\phi_e = m_1 B(t)$ and $\phi_b = m_2 E(t)$. Then based on the effectiveness of the intervention as illustrated by the literature, we set $m_1 = .25$ and $m_2 = .5$. The lower value of m_1 is due to the lack of support that is given to the victim, as in mediation. The level of accountability that is expected of the bully elevates the value of m_2 [12]. The resulting dynamic is shown in Figure 18. After 100 days the population of permanently immune individuals is just under 50%. The highest that the exposed and non-bullying population reach is 10%, while the susceptible population does not decrease as rapidly at the onset of bullying behavior. Instead it decreases to just under 40% and then the further decrease can be attributed to intervention rather than bullying.

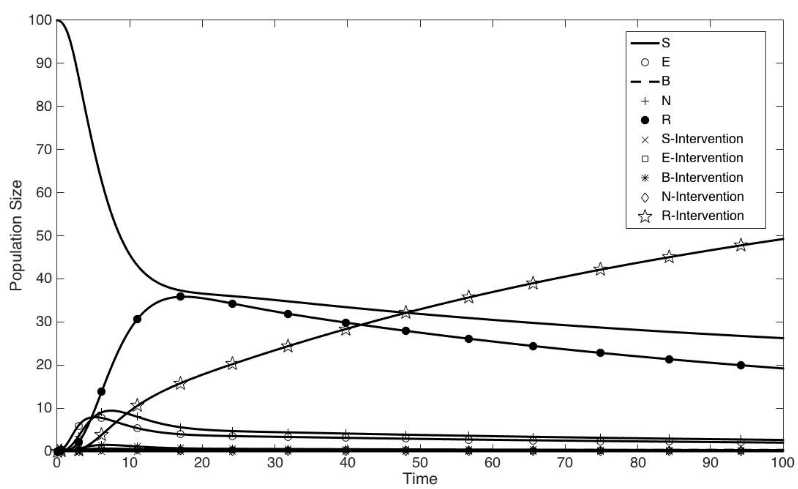


Figure 18: Numerical simulation of an implementation of the Restorative Justice Strategy that assumes that 50% of the bully-victim interactions motivated by a bully are mediated and 25% of the interactions motivated by a victim are mediated. Also we assume that 50% of those interventions will not result in a satisfactory compromise for each individuals. After fifty days the population is 49% recovered.

5.3.5 Support Group Method

The Support Group Method is the third in the set of interventions that hinge on positive interactions between the bully and the victim. This method focuses on providing substantial support to the victim as the mediation occurs. In (2) let ϕ_s, ϕ_n, and $\phi_r = 0$ and θ_s, θ_e, θ_b, and $\theta_n = .5$. Then as in the other mediation methods, let $\phi_e = m_1 B(t)$ and $\phi_b = m_2 E(t)$. Then based on the effectiveness of the intervention as illustrated by the literature, we set $m_1 = .5$ and $m_2 = .25$. The lower value of m_2 reflects the no-blame approach that characterizes this approach. The increased level of support given to the victim elevates the value of m_1 [12]. The resulting dynamic is shown in Figure 19. Under this intervention the permanently immune population has reached around 57% after 100 days. Similar to many of the other interventions the maximum populations of exposed and non-bullying individuals reaches 10% after 5 days. Also, as with other mediation based methods, the susceptible population is not fully impacted at the onset of the bullying behavior, instead it drops to 30% and then further decrease is due to intervention.

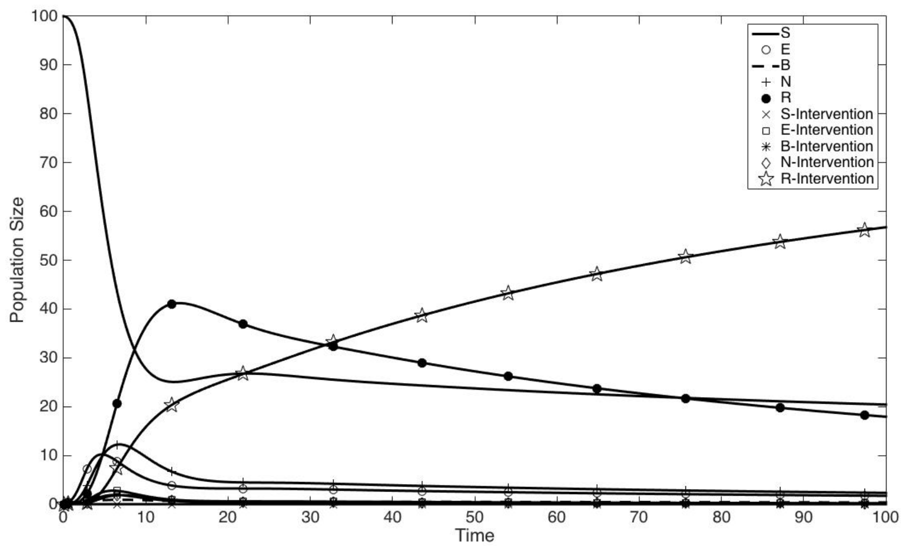


Figure 19: Numerical simulation of an implementation of the Support Group Method that assumes that 25% of the bully-victim interactions motivated by a bully are mediated and 50% of the interactions motivated by a victim are mediated. Also we assume that 50% of those interventions will not result in a satisfactory compromise for each individuals. After fifty days the population is 57% recovered.

5.3.6 The Method of Shared Concern

The final mediation based method is the Method of Shared Concern. While this is also a no blame approach the preparation before the final mediation generates significant accountability in the bully. The whole community actually becomes accountable while also becoming the support for the victim. To reflect this in (2) let ϕ_s, ϕ_n, and $\phi_r = .01$ to represent a small proportion of the non-involved individuals participating and being affected by the intervention. Let θ_s, θ_e, θ_b, and $\theta_n = .5$. Then as in the other mediation methods let $\phi_e = m_1 B(t)$ and $\phi_b = m_2 E(t)$. Then based on the effectiveness of the intervention as illustrated by the literature, we set $m_1 = .5$ and $m_2 = .5$ [12]. The resulting dynamic is shown in Figure 20. The Method of Shared Concern intervention results in a permanently immune population of 53% after 100 days. Similar to many of the other interventions the maximum populations of exposed and non-bullying individuals reaches just under 10% after 5 days. Also, as with other three mediation based methods, the susceptible population is not fully impacted at the onset of the bullying behavior, instead it drops to 40% and then further decrease is due to intervention.

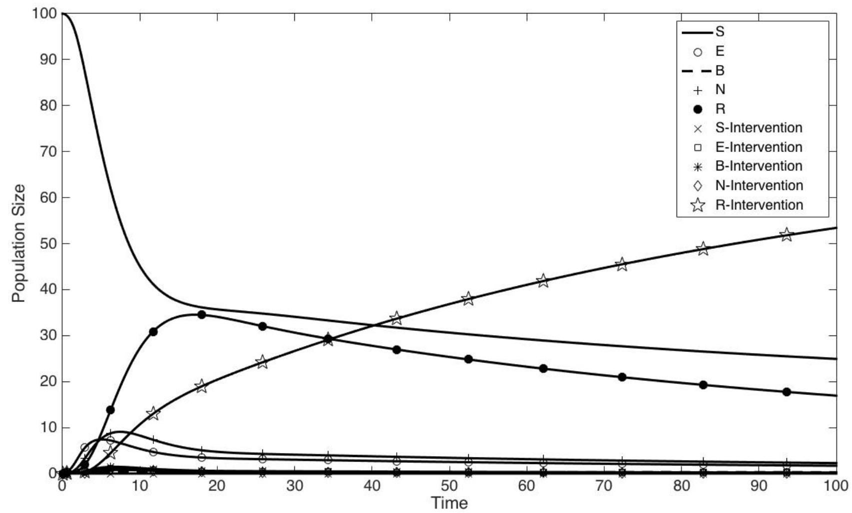


Figure 20: Numerical simulation of an implementation of the Method of Shared Concern intervention that assumes that 50% of the bully-victim interactions are mediated and that 1% of the rest of the community is involved in the mediation. Also we assume that 50% of those interventions will not result in a satisfactory compromise for each individuals. After fifty days the population is 60% recovered.

6 Conclusions

The SEBNR Bullying model (1) and the associated Intervention model (2) can be used to examine bullying in schools. Through the SEBNR model we have shown that bullying behaviors can and will persist in a population under varying parameter values. Our calculation and analysis of

$$R_0 = \frac{S_0 \beta p}{\alpha}$$

reveals that the spread of bullying in a population of size S_0 is determined by three parameters, β, the rate of bullying, p, the probability of becoming a bully after being bullied, and α, the recovery rate of bullies. A sensitivity analysis of the three parameters shows that for β and p the sensitivity index is equal to 1, while for α the index is equal to -1, making R_0 equally sensitive to changes in each of the parameters. For sufficiently large population sizes it becomes less feasible that the parameter values would be such that $R_0 < 1$. Thus, for realistic parameter values, that the system will reach an endemic equilibrium.

The inevitability of the spread of bullying motivates an analysis of the Intervention model. Through this model we analyzed six different intervention strategies identified by Ken Rigby as commonly used in schools. Each of the six strategies, (Traditional Disciplinary Approach, Strengthening the Victim, Mediation, Restorative Justice, Support Group Method, and Method of Shared Concern) gives rise to an Intervention sub-model. The Traditional Approach is by far the most common intervention strategy, yet by our numerical simulations, it is the least effective both in terms of the long and short term behavior of the system. Rampant bullying occurs at the introduction of a small number of bullies, and it takes the population a long time to recover. In contrast, the other five, less common intervention strategies all have better long term outcomes than that of the Traditional Approach.

Under the assumptions that were made about effectiveness of each treatment, the two strategies that generate the best long term behavior are Strengthening the Victim and the Method of Shared Concern. Each boasts significantly faster recovery than that of the Traditional Approach. The difference between the short term behaviors however is significant. The Strengthening the Victim approach does still have frequent bullying that occurs after the introduction of bullies into the system, whereas the Method of Shared Concern protects the population from bullying behaviors more effectively.

When each intervention is analyzed in context with the social situation the assumptions made about the two approaches come to light. It is feasible that a school intervention strategy could reach 50% of the potential victims, as students could be contacted en mass, and only count on 50% of those interventions to be successful throughout each stage of bullying behaviors. In fact one could reach nearly all the potential victims with the Strengthening the Victim approach, implying high ϕ values, and would only need to count on a small proportion of the interventions to be viable, implying low θ values, and the numerical simulations would still project positive long term behavior of the system. In contrast, the Method of Shared Concern is highly dependent on the mediation between the bully and the victim, which is a resource heavy intervention. To be successful it requires time, training and high levels of involvement of peers and adults. As a result the assumptions of effectiveness are feasible for a school intervention strategy, but at a much higher cost than that of the assumptions placed on the Strengthening the Victim approach. Schools should explore how they use their resources to intervene in bullying behaviors, and consider that while some of the alternative methods of intervention, especially the Strengthening the Victim approach, may not be a typical form of justice, they may be more effective for eliminating the bullying epidemic within their walls.

7 Acknowledgements

This research was made possible by the Student Faculty Collaborative Grant through the Dean's Office at Colorado College.

A Appendix

Theorem A.1. *Let* $F : \mathbb{R}_+^{n+1} \to \mathbb{R}^n$*, be locally Lipschitz*

$$F(t,x) = (F_1(t,x), \dots F_n(t,x)), \qquad x = (x_1, \dots, x_n)$$

and satisfy

$$F_j(t,x) \geq 0 \quad \textit{whenever} \quad t \geq 0, \qquad x \in \mathbb{R}_+^n, \qquad x_j = 0$$

Then, for every initial condition $x^0 \in \mathbb{R}_+^n$ *there exists a unique solution of* $x' = F(t,x)$*, which is defined on some interval* $[0,b)$*,* $b > 0$*. If* $b < \infty$*, then*

$$\limsup_{t \nearrow b} \sum_{j=1}^{n} x_j(t) = \infty.$$

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