COMPUTING DIMENSION OF LINEAR SYSTEMS ON GRAPHS

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Abstract

In this paper, we describe a method for computing the dimension of linear systems on a graph, which are related to linear systems of divisors on tropical curves. Tropical geometry is a discrete version of algebraic geometry, where a tropical curve can be represented by a metric graph. By reducing an algebraic curve to a graph, computing the dimension of a linear system can be thought of as a geometric problem. Specifically, we can compute the dimension of the linear system as the distance to a surface in n-dimensional space using the taxi-cab metric. Finally, we present examples of computing such dimensions of linear systems for divisors on 2-vertex and 3-vertex graphs.

Contents

1. Introduction

The dimension of linear systems associated with integer and real divisors on graphs has been defined in the work of Baker and Norine, and James and Miranda. Matthew Baker and Serguei Norine developed theory of integer divisors and linear systems on graphs as an analog to divisors and linear systems on algebraic curves [1]. Work by Rodney James and Rick Miranda generalized the concepts of linear systems and divisors to non-negative real-valued functions [4]. This concept of dimension of these linear systems associated with divisors on graphs has been used in a volume proof by An, Baker, Kuperberg, and Shokrieh [9] of Kirkhoff's Tree Theorem concerning the number of spanning trees of G.

Divisors on graphs are related to divisors on algebraic curves, since "tropicalizing" an algebraic curve results in a tropical curve which resembles a metric graph. Divisors on algebraic curves are zeros and poles of polynomials and thus are integers. However, the integers do not have multiplicative inverses, and the method we describe for computing dimension makes use of multiplicative inverses. The divisors we consider are real-valued since this enables the interesting geometric interpretation of dimension of a linear system on an n-vertex graph G as the taxi-cab distance to a solid region in $\mathbb{R}^n$ that contains divisor with empty linear systems. Note, however, that real-valued divisors are a generalization of integer-valued divisors; for integer-divisors, the points separating empty and non-empty linear systems form a lattice in $\mathbb{R}^n$ and this lattice is contained within the corresponding surface for real-divisors of the same graph.

The first three sections of this paper will introduce and develop necessary terminology, definitions, and formulas. The next three sections will present a method for calculating these dimensions for n-vertex graphs, and provide detailed examples applying this method for 2-vertex and 3-vertex graphs. The final section attempts to provide some explanation of the relationship between divisors on algebraic curves and divisors on metric graphs, by summarizing some of the relevant theory.

2. Divisors and Linear Systems on Graphs

Let G be a finite, connected graph without loops or multiple edges. The vertex set of G is $\{v_0, v_1, ...v_n\}$. Each edge of G is assigned a weight that is a positive element of R, where R a subring of real numbers $\mathbb{R}$. The weight of an edge connecting v_i and v_j is $w_{ij} = w_{ji}$. If there is no edge connecting the vertices i and j, then $w_{ij} = w_{ji} = 0$. The degree of a vertex is defined as the sum of the edge-weights of incident edges,

$$\deg(v_j) = \sum_{i \neq j} w_{ij}$$

The edge-weighted Laplacian matrix Δ is defined as the $n+1$ by $n+1$ matrix with vertex degree on the diagonal and ij and ji off-diagonal entries are the negative corresponding edge-weight values,

$$\Delta = \begin{cases} deg(v_i) & \text{if } i = j \\ -w_{ij} & \text{if } i \neq j \end{cases}$$

The edge-weighted reduced Laplacian Δ_0 is the Laplacian matrix without the first row and first column which are associated with the sink vertex v_0. The reduced Laplacian is a monotone, $n \times n$ matrix with full rank, meaning it has a nonnegative inverse [7] and is defined as,

$$\Delta_r = \begin{pmatrix} \deg(v_1) & -w_{12} & ... & -w_{1n} \\ -w_{12} & \deg(v_2) & ... & -w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -w_{1n} & -w_{2n} & ... & \deg(v_n) \end{pmatrix}$$

As defined in [2], an R-divisor D on G is a function $D : V \to R$. Divisors resemble vectors and can be written similarly in the convenient form of tuples or vectors of length $n+1$, where placement is indicative of association with a given vertex. Divisors are defined as a formal sum of values $d_i \in R$ over the vertices,

$$D = \sum_{i=0}^{n} d_i \cdot v_i$$

The set of R-divisors D on G is denoted by $\text{Div}(G)$. The degree of a divisor is defined as,

$$\deg(D) = \sum_{i=0}^{n} d_i$$

For any $x \in R$, $D > x$ means that $d_i > x$ for all $i = 0, 1, ...n$. We say $D > D'$ if $d_i > d'_i$ for $i = 0, 1, ...n$. A divisor is said to be effective if $\lceil D \rceil \geq 0$ i.e. $D > -1$. Equivalently, if R is the integer subring of $\mathbb{R}$ then D is said to be effective if $D \geq 0$.

One special group of divisors are called the $\mathcal{H}$-divisors. The $\mathcal{H}$-divisors are the columns of the edge-weighted Laplacian matrix. For $j = 0, 1, ...n$ the divisor $\mathcal{H}_j$ is defined as,

$$\mathcal{H}_j = \begin{cases} \deg(v_j) & \text{if } v_i \neq v_j \\ -w_{ij} & \text{otherwise} \end{cases}$$

A $\mathbb{Z}$-linear combination of $\mathcal{H}$-divisors is called a principle divisor $\mathcal{P}$. The set of principle divisors is the set of all integer linear combinations of the $\mathcal{H}$-divisors,

$$\mathrm{PDiv}(G) = \{D \in \mathrm{Div}(G) \mid D = z_0\mathcal{H}_0 + z_1\mathcal{H}_1 + ...z_{n-1}\mathcal{H}_{n-1} \text{ where } z_i \in \mathbb{Z}\}$$

Note that all principle divisors have degree zero.

Two divisors D and $D' \in \mathrm{Div}(G)$ are said to be linearly equivalent if and only if they differ by a principle divisor, i.e. $D - D' \in \mathrm{PDiv}(G)$, where subtraction is defined component wise. We use the notation $D \sim D'$ to indicate the linear equivalence of D and D'.

The linear system associated with a divisor D is the set of all effective divisors that are linearly equivalent to D. In mathematical notation,

$$\begin{aligned} |D| &= \{D' \in \mathrm{Div}(G) \mid \lceil D' \rceil \geq 0 \text{ and } D' \sim D\} \\ &= \{D' \in \mathrm{Div}(G) \mid D' > -1 \text{ and } D' \sim D\} \end{aligned}$$

If R is the subring of integers then we simply have,

$$|D| = \{D' \in \mathrm{Div}(G) \mid D' \geq 0 \text{ and } D' \sim D\}$$

3. The $\mathcal{N}$-Set

Another important group of divisors is the set of $\mathcal{N}$-divisors. Following the definition of this set given in [3], let $(j_0, j_1, ..., j_n)$ be a permutation of the vertices $(0, 1, ..., n)$ where we have fixed $j_0 = k$. There are $n!$ such permutations. A divisor $\nu \in \mathcal{N}$ can be calculated for each permutation, where ν is defined as,

$$\nu_{j_l} = \begin{cases} -1 & \text{if } l = 0 \\ -1 + \displaystyle\sum_{i=0}^{l} w_{j_i j_l} & \text{if } l > 0 \end{cases}$$

The $\mathcal{N}$-set is generated by the ν-divisors, meaning $\mathcal{N}(G)$ is the set of all divisors linearly equivalent to these ν-divisors. Say there are s unique ν-divisors,

$$\mathcal{N}(G) = \{D \in \mathrm{Div}(G) \mid D \sim \nu_i \text{ for } i = 1, 2, ...s\}\}$$

Divisors and their linear systems can be visualized in $\mathbb{R}^{n+1}$-space. The $\mathcal{N}$-divisors are significant because they identify corners of cones delineating a boundary in $\mathbb{R}^{n+1}$ between divisors with empty and non-empty linear-systems. The dimension of the linear system associated with any given divisor D depends on the location of D in that space. More specifically, the dimension can be calculated as the taxi-cab distance from the region containing divisors with empty linear systems. If the divisor is already within the region of empty linear systems, then that taxi-cab distance is zero. As James and Miranda demonstrate in [4], the dimension of a linear system can thus be defined as,

$$\ell(D) = \min_{\nu \in \mathcal{N}} \left\{ \sum_{i=0}^{n} \max\{d_i - \nu_i, 0\} \right\}$$

Because the $\mathcal{N}$-divisors are translates of the ν-divisors by principle divisors, the surface of intersecting cones delineated by the $\mathcal{N}$-set is repetitive in nature. Thus the surface can be constructed as the pattern of a certain fundamental region infinitely repeated across space. The fundamental region is the centered on a special divisor called the canonical divisor, defined as,

$$K = \sum_{i=0}^{n} v_i \cdot (\deg(v_i) - 2)$$

Regarding the symmetry of the $\mathcal{N}(G)$-set,

(1) The set $\mathcal{N}(G)$ is symmetric with respect to K, i.e. $\eta \in \mathcal{N}(G)$ if and only if $K - \eta \in \mathcal{N}(G)$.
(2) For any divisor $D \in \mathrm{Div}(G)$, $|D| = \emptyset$ if and only if there is $\eta \in \mathcal{N}(G)$ such that $D \leq \eta$.

By definition, any pair of distinct $\eta \in \mathcal{N}(G)$ differ by a principal divisor. Considering three-vertex graphs, the cross-product of the differences between any two pairs of $\eta \in \mathcal{N}$ is a cross product between two principal divisors,

$$\mathcal{P}_a = \begin{bmatrix} z_a1(w_{01} + w_{02}) - z_a2w_{01} \\ -z_a1w_{01} + z_a2(w_{01} + w_{12}) \\ -z_a1w_{02} - z_a2w_{12} \end{bmatrix} \text{ and } \mathcal{P}_b = \begin{bmatrix} z_b1(w_{01} + w_{02}) - z_b2w_{01} \\ -z_b1w_{01} + z_b2(w_{01} + w_{12}) \\ -z_b1w_{02} - z_b2w_{12} \end{bmatrix}$$

By carrying out the calculation of the determinant for any two principal divisor $\mathcal{P}_1, \mathcal{P}_2 \in \mathrm{PDiv}(G)$,

$$\mathcal{P}_a \times \mathcal{P}_b = \begin{vmatrix} i & j & k \\ z_{a1}(w_{01} + w_{02}) - z_{a2}w_{01} & -z_{a1}w_{01} + z_{a2}(w_{01} + w_{12}) & -z_{a1}w_{02} - z_{a2}w_{12} \\ z_{b1}(w_{01} + w_{02}) - z_{b2}w_{01} & -z_{b1}w_{01} + z_{b2}(w_{01} + w_{12}) & -z_{b1}w_{02} - z_{b2}w_{12} \end{vmatrix}$$

The resulting vector can be simplified to,

$$\mathcal{P}_a \times \mathcal{P}_b = r(1,1,1) \text{ where } r = (z_{a1}z_{b2} - z_{a2}z_{b1})(w_{01}w_{02} + w_{12}w_{01} + w_{12}w_{02})$$

Because of this symmetry and orientation of the surface generated by $\mathcal{N}$, $\mathcal{P} \in \mathrm{PDiv}(G)$ is always perpendicular to the line $K + t(1,1,1)$.

4. Reduced Divisors

Divisors that are linearly equivalent have equivalent relationships to the repeated pattern of the surface constructed from the $\mathcal{N}$-set in terms of distances to different faces of the cones. Thus, any given divisor D can be reduced to a linearly equivalent, unique, "reduced" divisor D_r that lies within the so-called fundamental region, i.e.,

$$D_r = D + \mathcal{P}$$

In [4] James and Miranda prove the uniqueness and existence of such reduced divisors, defining a reduced divisor as follows:

Let $V_0 = V - \{v_0\}$. A divisor D is said to be reduced if and only if,

(1) $D(v) > -1$ for each $v \in V_0$
(2) for every $I \subset \{1, 2, ..., n\}$ there is a $v \in V_0$ such that,

$$(D - \sum_{j \in I} H_j)(v) \leq -1$$

By this definition we see that a reduced divisor D is effective but sufficiently close to the boundary. Specifically, a reduced divisor is close enough to the boundary that subtracting one of the smallest principal divisors will make one of the components less than negative one.

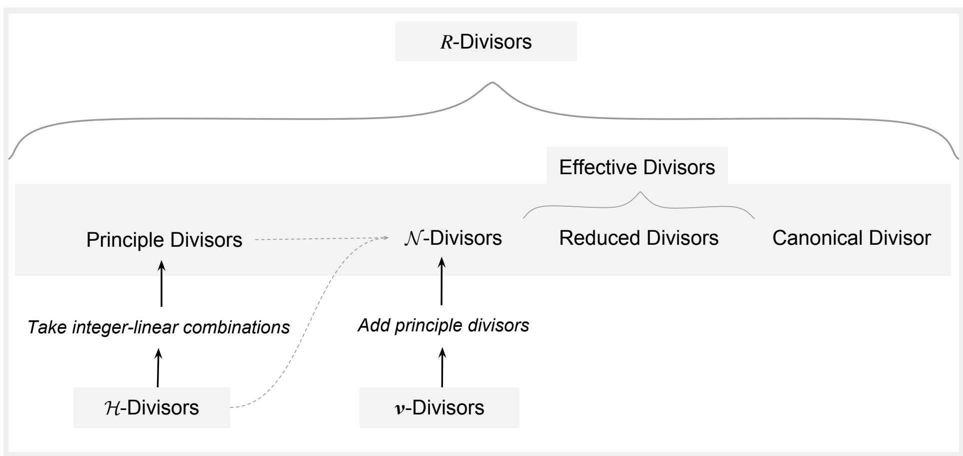


FIGURE 1. There are many different special subsets of the R-divisors. This diagram illustrates how these different types of divisors are related to one another. Arrows indicate a subset of divisors are used to generate a larger set. Brackets indicate a certain group of divisors is a subgroup of another group.

Because of the symmetry of the surface generated by $\mathcal{N}$, there is a certain vector $\mathcal{P}'$ that represents displacement of any divisor D from nearest point on the line $K + t\mathbf{v}$,

$$\mathcal{P}' = K + t\mathbf{v} - D \text{ where } t \text{ is given by solving } \frac{d}{dt}\Big(\sum_{i=0}^{n}(D(v_i) - K(v_i) + t)^2\Big) = 0$$

$$\mathcal{P}' = s\mathcal{P} \text{ where } \frac{1}{2} < s < \frac{3}{2} \text{ and } s \in \mathbb{R}$$

If s is less than one-half, then either there must be a smaller $\mathcal{P}$ that would make the equality true, or D must already lie within the fundamental region meaning $c_i = 0$. If s is larger than three-halves, there exists another larger principal divisor that should be used instead.

Define a function $\phi : \mathrm{Div}(G) \to R^n$ that takes the part of the divisor associated with $v_1, ..., v_n$ but not v_0 where $\phi(D) = (D(v_1), ..., D(v_n))$. The full Laplacian matrix is not invertible because of linear dependence of the columns, but the reduced Laplacian is. Thus we want to derive relationships involving the reduced Laplacian Δ_r rather than Δ. This is where the ϕ-function is useful.

$$\Delta \mathbf{c} = \begin{bmatrix} \mathcal{H}_0 & \mathcal{H}_1 & \dots & \mathcal{H}_n \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \deg(v_0)c_0 - w_{01}c_1 - \dots - w_{0n}c_n \\ -w_{01}c_0 + \deg(v_1)c_1 - \dots - w_{1n}c_n \\ \vdots \quad\quad \vdots \quad\quad \ddots \quad\quad \vdots \\ -w_{0n}c_0 - w_{1n}c_1 - \dots + \deg(v_n)c_n \end{bmatrix}$$

$$\mathcal{P} = \begin{bmatrix} \mathcal{H}_1 & \dots & \mathcal{H}_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} -w_{01}c_1 - w_{02}c_2 - \dots - w_{0n}c_n \\ \deg(v_1)c_1 - w_{12}c_2 - \dots - w_{1n}c_n \\ \vdots \quad\quad \vdots \quad\quad \ddots \quad\quad \vdots \\ -w_{1n}c_1 - w_{2n}c_2 - \dots + \deg(v_n)c_n \end{bmatrix}$$

$$\phi(\mathcal{P}) = \Delta_r \mathbf{c}_r = \begin{bmatrix} \phi(\mathcal{H}_1) & \dots & \phi(\mathcal{H}_n) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \deg(v_1)c_1 - w_{12}c_2 - \dots - w_{1n}c_n \\ -w_{12}c_1 + \deg(v_2)c_2 - \dots - w_{2n}c_n \\ \vdots \quad\quad \vdots \quad\quad \ddots \quad\quad \vdots \\ -w_{1n}c_1 - w_{2n}c_2 - \dots + \deg(v_n)c_n \end{bmatrix}$$

We can then derive the following relationships, which form the foundation of a conjectured method to compute reduced divisors,

$$\mathcal{P}' = s\mathcal{P} \quad \longrightarrow \quad \phi(\mathcal{P}) = \phi(\tfrac{1}{s}\mathcal{P}')$$

$$\phi(\mathcal{P}) = \Delta_r \mathbf{c}_r \quad \longrightarrow \quad \mathbf{c}_r = \tfrac{1}{s}\Delta_r^{-1}\phi(\mathcal{P}')$$

$$D_r = D + \mathcal{P} \quad \longrightarrow \quad \phi(D_r) = \phi(D) + \phi(\mathcal{P}) = \phi(D) + \Delta_r \mathbf{c}_r$$

$$\mathcal{P}(v_0) = -\textstyle\sum_{i=1}^n \phi(\mathcal{P}(v_i)) = -\sum_{i=1}^n \Delta_r \mathbf{c}_r(v_i) = -w_{01}c_1 - w_{02}c_2 - \dots - w_{0n}c_n = -\sum_{i=1}^n w_{0i}c_i$$

$$D_r(v_0) = D(v_0) - \mathcal{P}_0 = D(v_0) - \textstyle\sum_{i=1}^n \Delta_r \mathbf{c}_r(v_i) = D(v_0) - \sum_{i=1}^n w_{0i}c_i$$

In practice, we can find $\mathbf{c}_r$ by plotting $\frac{1}{s}\Delta_r^{-1}\phi(\mathcal{P}')(v_i)$ where $\frac{1}{2} < s < \frac{3}{2}$ for each v_i, and identifying the integer of the smallest magnitude which falls on the curve within the bounds on s. More formally, we could find c_i by writing an algorithm based on taking the minimum of the absolute value of the integer part of the function and then obtaining the correct sign by computing whether the $\Delta_r^{-1}\mathcal{P}'(v_i)$ is greater than or less than zero. Once $\mathbf{c}_r$ is found, the reduced divisor D_r can easily be calculated.

5. Dimensions of Linear Systems on 2-vertex Graphs

Let G be a finite, connected graph without loops or multiple edges, with two vertices and edge-weight $w > 0$. The Laplacian of G is,

$$\Delta = \begin{bmatrix} w & -w \\ -w & w \end{bmatrix}$$

The principle divisors are therefore $\mathrm{PDiv}(G) = \{(nw, -nw) \mid n \in \mathbb{Z}\}$. The linear system associated with a divisor $D \in \mathrm{Div}(G)$ is,

$$|(a,b)| = \{(a+nw, b-nw) \mid n \in \mathbb{Z}, a+nw > -1, b-nw > -1\}$$

Claim: $|(a,b)| \neq \emptyset$ if and only if $\lceil \frac{a+1}{w} \rceil + \lceil \frac{b+1}{w} \rceil \geq 2$

Proof: $|(a,b)| \neq \emptyset$ means there is at least one effective divisor linearly equivalent to (a,b), i.e. $(a+nw, b-nw)$ where $a+nw > -1$ and $b-nw > -1$. By definition of ceiling and floor functions, $n = \lfloor \frac{b+1}{w} - \epsilon \rfloor < \frac{b+1}{w} \leq \lceil \frac{b+1}{w} \rceil = n+1$, where $\epsilon = 1$ if $\frac{b+1}{w} \in \mathbb{Z}$ and $\epsilon = 0$ otherwise. By this inequality, $n < \frac{b+1}{w} \longrightarrow b - nw > -1$. Since $\frac{a+1}{w} + 1 > \lceil \frac{a+1}{w} \rceil$,

$$\left(\tfrac{a+1}{w} + 1\right) + \left(n+1\right) > \lceil \tfrac{a+1}{w} \rceil + \lceil \tfrac{b+1}{w} \rceil \geq 2$$

Then $\frac{a+1}{w} > -n \longrightarrow a + nw > -1$. Thus, if $\lceil \frac{a+1}{p} \rceil + \lceil \frac{b+1}{p} \rceil \geq 2$ there exists $n \in \mathbb{Z}$ such that both inequalities $a + nw > -1$ and $b - nw > -1$ are true, i.e. the linear system associated with (a,b) is non-empty. To prove the other direction of the implication made in the claim, consider,

$$\lceil \tfrac{a+1}{w} \rceil + \lceil \tfrac{b+1}{w} \rceil < 2$$

$$\lceil \tfrac{a+1}{w} \rceil < 2 - \lceil \tfrac{b+1}{w} \rceil$$

$$\tfrac{a+1}{w} \leq \lceil \tfrac{a+1}{w} \rceil \text{ and } \tfrac{b+1}{w} \leq \lceil \tfrac{b+1}{w} \rceil$$

$$\tfrac{a+1}{w} \leq \lceil \tfrac{a+1}{w} \rceil < 2 - \lceil \tfrac{b+1}{w} \rceil \leq 2 - \tfrac{b+1}{w}$$

$$\tfrac{a+1}{w} + n \leq \lceil \tfrac{a+1}{w} \rceil + n < 2 - \lceil \tfrac{b+1}{w} \rceil + n \leq 2 + n - \tfrac{b+1}{w}$$

By way of contradiction, assume that the linear system associated with (a, b) is non-empty and thus there is at least one divisor such that $a + nw > -1$ holds and $b - nw > -1$ is also true. Then,

$$a + nw > -1 \longrightarrow \frac{a+1}{w} + n > 0$$

$$0 < \frac{a+1}{w} + n \leq \left\lceil \frac{a+1}{w} \right\rceil + n < 2 - \left\lceil \frac{b+1}{w} \right\rceil + n \leq 2 - \frac{b+1}{w} + n$$

$$\frac{b+1}{w} - n > 0 \longrightarrow n - \frac{b+1}{w} < 0$$

$$0 < \frac{a+1}{w} + n \leq \left\lceil \frac{a+1}{w} \right\rceil + n < 2 - \left\lceil \frac{b+1}{w} \right\rceil + n \leq 2 + n - \frac{b+1}{w} < 2$$

$$0 < \left\lceil \frac{a+1}{w} \right\rceil + n < 2 - \left\lceil \frac{b+1}{w} \right\rceil + n < 2$$

$$\left\lceil \frac{a+1}{w} \right\rceil + n \in \mathbb{Z} \text{ and } \left\lceil \frac{b+1}{w} \right\rceil + n \in \mathbb{Z}$$

There is only one integer between zero and two, but $\left\lceil \frac{a+1}{w} \right\rceil + n \neq 2 - \left\lceil \frac{b+1}{w} \right\rceil + n$. Therefore, if $\left\lceil \frac{a+1}{w} \right\rceil + \left\lceil \frac{b+1}{w} \right\rceil < 2$ then there is no $n \in \mathbb{Z}$ such that both $a + nw > -1$ and $b - nw > -1$ can be true. Thus, the linear system associated with (a, b) is empty if $\left\lceil \frac{a+1}{w} \right\rceil + \left\lceil \frac{b+1}{w} \right\rceil < 2$.

Let $\varphi_w(x) = \left\lfloor \frac{(x+1)}{w} \right\rfloor$. Note that $\left\lceil \frac{a+1}{w} \right\rceil + \left\lceil \frac{b+1}{w} \right\rceil \geq 2$ is equivalent to saying that $\left\lfloor \frac{a+1}{w} \right\rfloor + \left\lfloor \frac{b+1}{w} \right\rfloor \geq 0$. If the linear system is empty, then the dimension is defined to be zero. The dimension of the linear system associated with a divisor $D = (a, b)$ on a 2-vertex graph G can be calculated by the following formula, derived by James and Miranda in [2],

$$\ell((a,b)) = \begin{cases} 0 & \text{if } \varphi(a) + \varphi(b) < 0 \\ \min\{a + 1 - w\varphi_w(a), b + 1 - w\varphi_w(b)\} & \text{if } \varphi(a) + \varphi(b) = 0 \\ a + b - w + 2 & \text{if } \varphi(a) + \varphi(b) > 0 \end{cases}$$

Suppose we have the two-vertex graph G with edge weight $w = 4.6$. Then $\Delta_r = 4.6$ and so $\Delta_r^{-1} = \frac{10}{46}$. Consider the divisor $D = (10.4, -7)$. We can calculate the dimension of the linear system associated with D via the formula given above,

$$\varphi(10.4) + \varphi(-7) = \lfloor (10.4 + 1)/4.6 \rfloor + \lfloor (-7 + 1)/4.6 \rfloor = \lfloor 2.48 \rfloor + \lfloor -1.30 \rfloor = 0$$

$$\begin{aligned} \ell((10.4, -7)) &= \min\{10.4 + 1 - 4.6 \lfloor (10.4 + 1)/4.6 \rfloor, -7 + 1 - 4.6 \lfloor (-7 + 1)/4.6 \rfloor\} \\ &= \min\{11.4 - 4.6(2), -7 + 1 - 4.6(-2)\} = \min\{2.2, 3.2\} = 2.2 \end{aligned}$$

Alternatively, reducing a divisor and calculating minimum distance of the reduced divisor from the $\eta \in \mathcal{N}$ that define the fundamental region gives the taxi-cab distance to the divisor surface and the value for the dimension of the linear system.

$$K = (\deg(v_0) - 2, \deg(v_1) - 2) = (2.6, 2.6)$$

$$\frac{d}{dt}\Big((-7.8 + t)^2 + (9.6 + t)^2\Big) = \frac{d}{dt}\Big(153. + 3.6t + 2t^2\Big) = 3.6 + 4t = 0 \longrightarrow t = -0.9$$

$$\mathcal{P}' = K + t(1,1) - D = (2.6, 2.6) + (-0.9, -0.9) - (10.4, -7) = (-8.7, 8.7)$$

$$\frac{1}{s}\Delta_r^{-1}\phi(\mathcal{P}') = \frac{1}{s}\frac{10}{46}(8.7) = 1.8913\frac{1}{s}$$

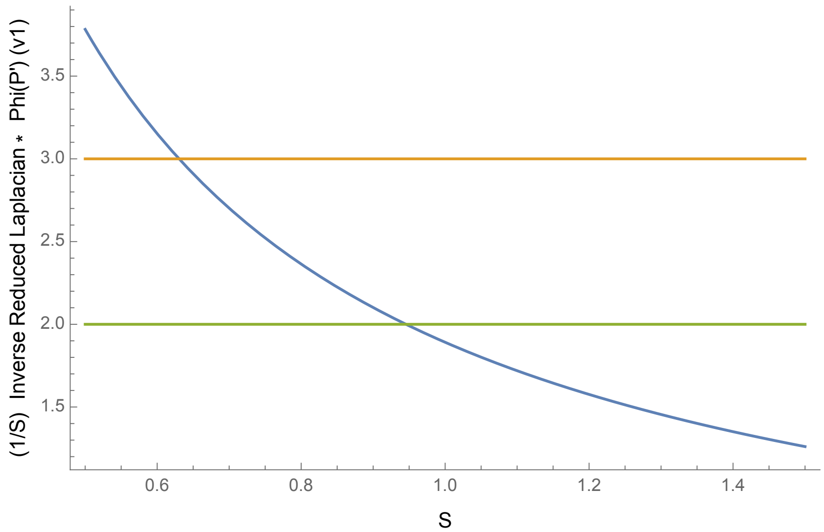


FIGURE 2. This plot shows the function $1.8913/s$ where $1/2 < s < 3/2$, with horizontal lines identifying where the function possesses integer values.

From the plot we can see that the integer with smallest magnitude, $\mathbf{c}_r$, is 2. Then,

$$\phi(D_r) = \phi(D) + \Delta_r \mathbf{c}_r = -7 + (4.6)(2) = 2.2$$

$$D_r(v_0) = D(v_0) - \textstyle\sum_{i=1}^{n} \Delta_r \mathbf{c}_r(v_i) = 10.4 - (4.6)(2) = 1.2$$

$$D_r = (1.2, 2.2)$$

To calculate the dimension of D, we need $\eta_1, \eta_2 \in \mathcal{N}(G)$ defining the fundamental region,

$$\eta_1 = (-1, 3.6) \text{ and } \eta_2 = (3.6, -1)$$

$$\begin{aligned}
\ell(D) &= \min_{\eta \in \mathcal{N}} \{\textstyle\sum_{i=0}^{n} \max\{d_i - \nu_i, 0\}\} \\
&= \min\{\max\{1.2 + 1, 0\} + \max\{2.2 - 3.6, 0\}, \max\{1.2 - 3.6, 0\} + \max\{2.2 + 1, 0\}\} \\
&= \min\{2.2, 3.2\} \\
&= 2.2
\end{aligned}$$

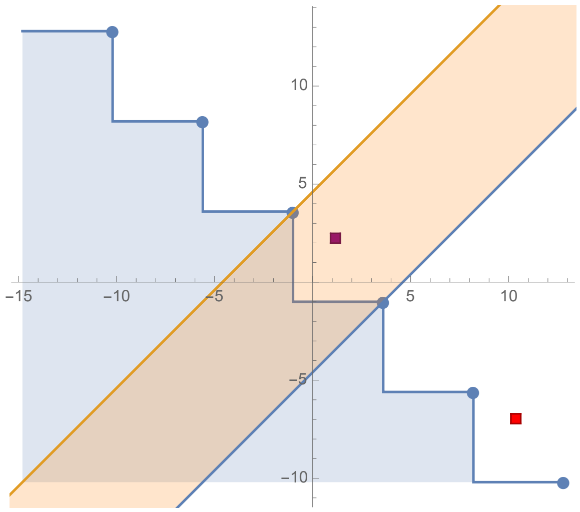


FIGURE 3. Possible divisors on the 2-vertex graph G with edge-weight $w = 4.6$. Divisors that fall in the shaded region behind the stair-step boundary have empty linear systems and dimension zero. The set of points on the corners of the stair-steps are in the $\mathcal{N}$-set. The red point is $D = (10.4, -7)$ and the purple point is the reduced divisor $D_r = D + \mathcal{P} = (1.2, 2.2)$.The orange-shaded region is the fundamental region.

6. Use of the Graph-theoretic Riemann-Roch Theorem

The three-part formula seen in the example for the 2-vertex graph above, arises from the graph-theoretic analog of the Riemann-Roch Theorem for algebraic curves. In [1], Baker and Norine derived this Riemann-Roch Theorem for integer divisors on graphs from the classical Riemann-Roch Theorem, and in [2] James and Miranda prove a Riemann-Roch theorem for real divisors on edge-weighted graphs. Gathmann and Kerber also present a Riemann-Roch theorem, specifically for tropical geometry in [10].

Theorem 6.1 (The Riemann-Roch Theorem for Graphs)**.** *For any divisor* $D \in \mathrm{Div}(G)$

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g$$

The genus g of the graph G is defined as,

$$g = \sum_{i<j} w_{ij} - n + 1$$

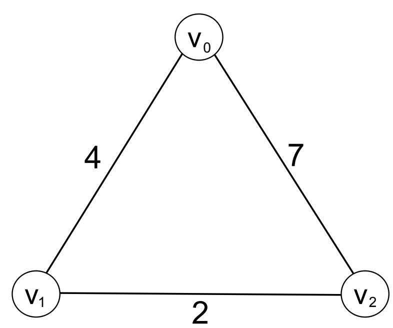
Note that by this definition the genus is not necessarily an integer, and furthermore, can be negative when the edge weights w_{ij} are sufficiently small.

By this theorem, the problem of calculating the dimension of a linear system associated with a divisor D on a 2-vertex graph can be split into three cases, based on whether the individual pieces $\ell(D)$ and $\ell(K - D)$ are non-zero. If $\ell(D) = 0$ and $\ell(K - D) = 0$ and we get the first line of the formula, if $\ell(D) \neq 0$ and $\ell(K - D) \neq 0$ then we get the second line, and if $\ell(D) \neq 0$ and $\ell(K - D) = 0$ then we get the third line of the formula. Each of these cases corresponds to certain categories of possible divisors, as can be seen in Figure 3.

The first line of the the formula corresponds to divisors in the shaded region below and to the left of $\mathcal{N}$-set points which delineate a stair-like boundary. These are non-effective divisors. The second line of the formula corresponds to divisors that lie close enough to the boundary that the minimum taxi-cab distance to the stair-step boundary is not one of the $\mathcal{N}$-set points at the corners of the steps. The third line of the formula corresponds to all other divisors. One of the advantages of this theorem in computing dimensions of linear systems associated with divisors is that it simplifies the computation; by the Riemann-Roch theorem for graphs, if $K - D \leq 0$ the dimension $\ell(D)$ is simply the $\deg(D) - g + 1$.

7. Dimensions of Linear Systems on 3-vertex Graphs

Let G be the following graph, which is connected without multiple edges or loops:



The Laplacian of this graph is: $\Delta = \begin{bmatrix} 11 & -4 & -7 \\ -4 & 6 & -2 \\ -7 & -2 & 9 \end{bmatrix}$

The reduced Laplacian is: $\Delta_r = \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}$

The inverse reduced Laplacian is: $\Delta_r^{-1} = \begin{bmatrix} \frac{9}{50} & \frac{1}{25} \\ \frac{1}{25} & \frac{3}{25} \end{bmatrix}$

The canonical divisor $K = (\deg(v_0) - 2, \deg(v_1) - 2, \deg(v_2) - 2) = (11 - 2, 6 - 2, 9 - 2) = (9, 4, 7)$

Let $j_0 = 0$, then the permutations (j_0, j_1, j_2) of the vertices $(0, 1, 2)$ are $(0, 1, 2)$ and $(0, 2, 1)$. By the formula for ν_{j_l} we get,

$\nu_1 = (-1, -1+w_{01}, -1+w_{02}+w_{12}) = (-1, 3, 8)$ and $\nu_2 = (-1, -1+w_{02}, -1+w_{01}+w_{12}) = (-1, 6, 5)$

From the Laplacian, $\mathcal{H}_1 = (11, -4, -7)$ and $\mathcal{H}_2 = (-4, 6, -2)$. The principal divisors are $\mathrm{PDiv}(G) = \{\mathcal{P} = z_1\mathcal{H}_1 + z_2\mathcal{H}_2 \mid z_1, z_2 \in \mathbb{Z}\}$. Thus we get the $\mathcal{N}$-set which defines the corners of cones forming the boundary surface,

$\mathcal{N}(G) = \{(-1+11n-4m, 3-4n+6m, 8-7n-2m), (-1+11n-4m, 6-4n+6m, 5-7n-2m) \mid n, m \in \mathbb{Z}\}$

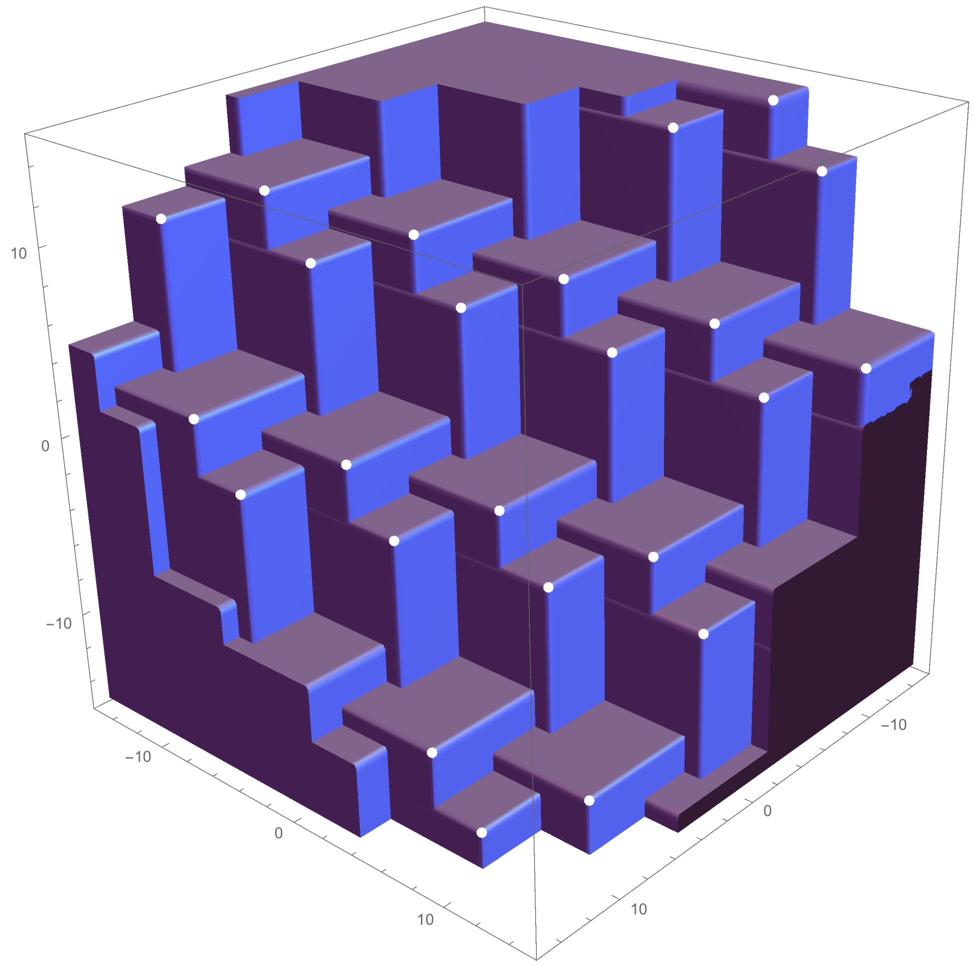


FIGURE 4. This graphic shows regions in $\mathbb{R}^{n+1}$ space of divisors with empty versus non-empty linear systems. Divisors with empty linear systems, and thus dimension zero, fall behind the surface in the solidly colored region. The points on the corners of the cones are a subgroup N of the infinite $\mathcal{N}$-set which correspond to the portion of $\mathbb{R}^{n+1}$ space being represented in this graphic.

Consider the divisor $D = (20, 15, -10)$.

$$\frac{d}{dt}\Big(\sum_{i=0}^{n}(K(v_i) + t - D(v_0))^2\Big) = \frac{d}{dt}\Big(531 - 10t + 3t^2\Big) = 6t - 10 = 0 \longrightarrow t = \frac{5}{3}$$

$$\mathcal{P}' = K + t(1,1,1) - D = (9,4,7) + (\tfrac{5}{3}, \tfrac{5}{3}, \tfrac{5}{3}) - (20, 15, -10) = (-\tfrac{28}{3}, -\tfrac{28}{3}, \tfrac{56}{3})$$

$$\mathbf{c}_r = \frac{1}{s}\Delta_r^{-1}\phi(\mathcal{P}') = \frac{1}{s}(-\tfrac{14}{15}, \tfrac{28}{15}), \text{ where } \tfrac{1}{2} < s < \tfrac{3}{2}$$

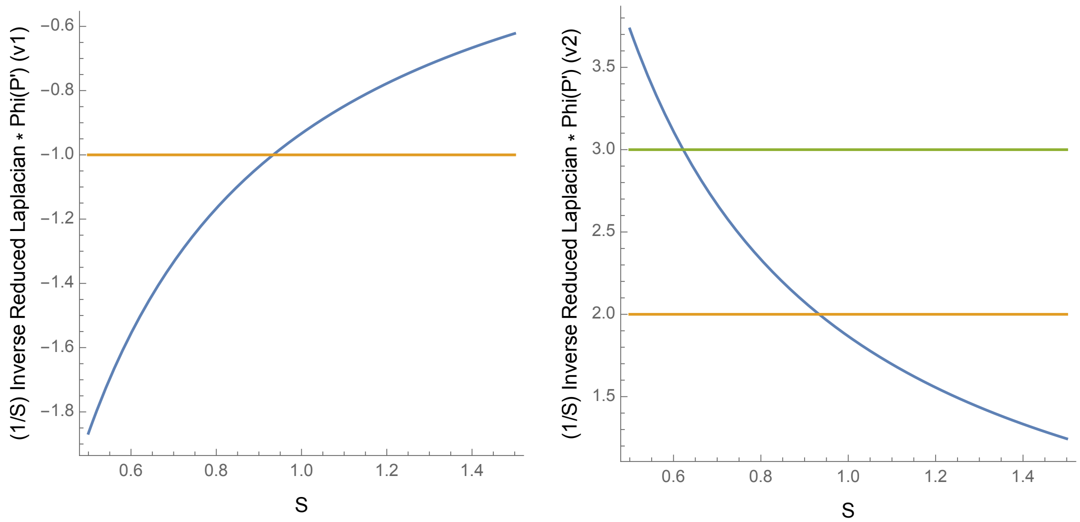


FIGURE 5. These plots show the functions $-14/15s$ and $28/15s$, for the range $1/2 < s < 3/2$. Horizontal lines indicate where the functions possesses integer values.

From the plots above, we can see that $\mathbf{c}_r = (-1, 2)$. Then,

$$\phi(D_r) = \phi(D) + \phi(\mathcal{P}) = \phi(D) + \Delta_r \mathbf{c}_r = \begin{bmatrix} 15 \\ -10 \end{bmatrix} + \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$$D_r(v_0) = D(v_0) - \mathcal{P}_0 = D(v_0) - \sum_{i=1}^{n} w_{0i} c_i = 20 - (4 \cdot (-1) + 7 \cdot 2) = 10$$

$$D_r = (10, 5, 10)$$

The $\eta \in N \subset \mathcal{N}(G)$ defining the fundamental region, i.e. with minimum distance to K are:

$$(3, 0, 7), (10, 2, -2), (6, 5, -1), (10, -1, 1), (-1, 3, 8), (-1, 6, 5)$$

The dimension of D then is the dimension of D_r which can be calculated via the taxi-cab metric,

$$\begin{aligned} \ell(D) &= \min_{\eta \in \mathcal{N}} \left\{ \sum_{i=0}^{n} \max\{D_r(v_i) - \eta(v_i), 0\} \right\} \\ &= \min \{15, ..., 15, 16\} = 15 \end{aligned}$$

8. Algebraic Curve, Tropical Curve, Metric Graph

This theory about divisors on graphs was originally derived as an analogy to the divisors on Riemann surfaces, which are one-dimensional complex manifolds. Essentially, a Reimann surface is a topological space that at each point, locally looks like the complex plane, and therefore one can define on a Riemann surface things that are definable on the complex plane, e.g. holomorphic functions and meromorphic functions. As described by Miranda in [6], a divisor on a Riemann surface X is a function $D : X \to \mathbb{Z}$ where the set of points $p \in X$ such that $D(p) \neq 0$ is a discrete subset of X. Intuitively, a divisor is a way of representing zeros and poles of a meromorphic function.

Following Imre Simon's development of tropical geometry as presented by Maclagan and Sturmfels in [5], the Tropical Semiring is $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$, where the operations of addition and multiplication are defined to be,

$$x \oplus y := \min(x, y) \qquad\qquad x \odot y := x + y$$

Addition is associative and commutative, and there exists an additive identity element. Multiplication is associative, and distributive laws for multiplication over addition hold. Division is defined as classical subtraction, but there is no tropical subtraction. All ring axioms are therefore satisfied except for the existence of an additive inverse, which is what makes the algebraic object $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ a semiring.

Let $x_1, x_2, ...x_n$ be variables representing elements in the tropical semiring. A tropical monomial is any product of these variables, for example,

$$x_3 \odot x_4 \odot x_3 \odot x_1 \odot x_2 \odot x_3 \odot x_4 = x_1 x_2 x_3^3 x_4^2$$

By classical arithmetic this would be a linear function with integer coefficients,

$$x_3 + x_4 + x_3 + x_1 + x_2 + x_3 + x_4 = x_1 + x_2 + 3x_3 + 2x_4$$

A tropical polynomial $p(x_1, ..., x_n)$ is a finite linear combination of tropical monomials, and represents a function $\mathbb{R}^n \to \mathbb{R}$. Having coefficients $a, b... \in \mathbb{R}$ and $i_1, j_1, ... \in \mathbb{Z}$,

$$p(x_1, ..., x_n) = a \odot x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \oplus b \odot x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \oplus \cdots$$

By classical arithmetic this is,

$$p(x_1, ..., x_n) = \min\{a + i_1 x_1 + i_2 x_2 + \cdots + i_n x_n, \quad b + j_1 x_1 + j_2 x_2 + \cdots + j_n x_n, \cdots\}$$

Thus a tropical polynomial is the minimum of a finite set of linear functions, and therefore a continuous, concave, and piecewise linear function.

The hypersurface $V(p)$ of a tropical polynomial p is set of all points in $\mathbb{R}^n$ at which the minimum of the finite set of linear functions given by p is attained at least twice, i.e. the set of points at which p is not linear or the set of roots of the polynomial p. Thus the curve $V(p)$ of the tropical polynomial p is a finite metric graph.

As described in [5], there exists a relation between algebraic curves on Riemann surfaces and tropical curves, via the transformation of an amoeba converging to its spine. Following Gathmann's explanation in [8], mapping a complex plane curve C on the open subset $(\mathbb{C}^*)^2$ to the real plane via the function $Log : (\mathbb{C}^*)^2 \to \mathbb{R}^2$ where $Log(z_1, z_2) \to (x_1, x_2) := (\log|z_1|, \log|z_2|)$ gives the amoeba $\mathcal{A}$. Every amoeba has a spine, which is essentially a tropical curve $V(p)$ contained in the amoeba.

A tropical curve $V(p)$ can be derived from an amoeba by taking a sequence of amoebas converging to their spine, shrinking the amoeba to zero width. To achieve this, suppose the map Log is given instead by,

$$Log(z_1, z_2) \to (x_1, x_2) := (-\log_t |z_1|, -\log_t |z_2|) = \big(-\frac{\log_t |z_1|}{\log t}, -\frac{\log_t |z_2|}{\log t}\big)$$

Taking the limit as t tends to zero gives the tropical curve or spine of the amoeba. The coefficients of the tropical polynomial p associated with the resulting tropical curve $V(p)$ are the zeros and poles of the coefficients of the algebraic curve C from which the amoeba was originally derived. Another method of deriving tropical curves from amoebas involves considering the asymptotic directions of the amoeba's tentacles. The amoeba has tentacles that converge to rays in $\mathbb{R}^2$. The union of these rays can be represented as a tropical plane curve $V(p)$.

9. Conclusion

Dimension of linear systems associated with divisors on metric n-vertex graphs can be calculated via the taxi-cab metric. Linear equivalence allows any given divisor D to be reduced to a unique reduced divisor D_r within a certain region in $\mathbb{R}^{n+1}$ with known bounds and symmetry, thus simplifying the problem of finding minimum distance over a set of points. This thesis conjectures about a method by which to do this, which involves finding a vector $\mathcal{P}'$ with minimum distance from D to the line $K + t\mathbf{v}$ where $\mathbf{v}$ is a unit vector, and then finding a principal divisor such that $\mathcal{P}' = s\mathcal{P}$ where $\frac{1}{2} < s < \frac{3}{2}$. Further research would include formal proof of the conjectured method for reducing a given divisor, and evaluation of the computational complexity of this method.

Simplification is a common strategy in mathematics, whether in simplifying systems or identifying special subgroups of mathematical objects, or reduce to a problem that has already been solved. Topicalization of algebraic curves is an example of this phenomenon. The hope is that by simplifying the algebraic curve to a discrete tropical curve, new relationships can be discovered, and insights can be gleaned about the original structure. In the case of divisors on graphs, particularly real-valued divisors and dimensions of their linear systems, there is much yet to be discovered.

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