An Introduction to Logic Algebras

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Abstract

We utilize the setting of Universal Algebra to introduce a new class of objects called *logics*, with the aim of generalizing the structure of familiar binary logic to a family of finite and countably infinite multi-valued logics. In Section 2, we explore several concepts and results parallel to those of more familiar algebraic structures and provide as an example an independent proof of the first isomorphism theorem for logics. In Section 3, we review some basic notions from Category Theory, which give us another lens through which to view logics, then prove several categorical results about them. In Section 4 we discuss implications of the idea of logics for various areas of mathematics, for the sake of brevity providing only a skeletal outline of what these might be, and in the last section we discuss the implications of logics on formal languages and natural deduction, providing a framework through which one might create generalized propositional logic.

Prologue : A Quick Introduction to Universal Algebra

It is not lost on most introductory algebra students that rings and groups share some common features (as do modules, semigroups, quasigroups, loops, lattices, boolean algebras and so on). The question then arises whether these algebraic structures all fall into a more abstract category of objects. Universal Algebra is the field that answers this question with a particular definition, and is a useful tool for proving general results (such as isomorphism theorems) about *all* such algebraic structures in one fell swoop.

Definition. An **n-ary operation** on a non-empty set A is any map from $A^n \to A$. All such operations are **finitary**. Any finite collection of operations is called a **family** of operations.

Definition. An **algebra** is an ordered pair $\mathcal{A} = (A, \mathcal{F})$ where A is a non-empty set, and $\mathcal{F}$ is a family of finitary operations on A. We call A the **universe** of $\mathcal{A}$.

Defintion. If two algebras have the same family $\mathcal{F}$, they are said to be of the same **type**.

Example A. G $= (G, \cdot, ^{-1}, e)$ where G is a set, $\cdot$ is a binary operation on G, $^{-1}$ is unary, and e is a nullary, such that the following familiar axioms hold,

G1 $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

G2 $x \cdot e = e \cdot x = x$

G3 $x \cdot x^{-1} = x^{-1} \cdot x = e$

is a type of algebra known as a group.

Example B. $(L, \vee, \wedge)$, where L is a set, $\vee$ and $\wedge$ are binary operations on L such that the following axioms hold, L1 :

(a) $x \vee y = y \vee x$

(b) $x \wedge y = y \wedge x$

L2 :

(a) $x \vee (y \vee z) = (x \vee y) \vee z$

(b) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

L3 :

(a) $x \vee x = x$

(b) $x \wedge x = x$

L4 :

(a) $x = x \vee (x \wedge y)$

(b) $x = x \wedge (x \vee y)$

is a type of algebra known as a lattice.

What what about things like homomorphisms, kernels, quotients? Universal Algebra provides us with generalized definitions of such concepts as well, and I will briefly enumerate several which will be relevant to our discourse later on.

Definition. Let $\mathcal{A}, \mathcal{B}$ be two algebras of the same type $\mathcal{F}$. Then a function $h : A \to B$ is a **homomorphism** from $\mathcal{A}$ to $\mathcal{B}$ if for every n-ary $f \in \mathcal{F}$, and for all $a_1, ..., a_n \in A$ we have,

$$h(f^{\mathcal{A}}(a_1, ..., a_n)) = f^{\mathcal{B}}(h(a_1), ..., h(a_n))$$

If such a homomorphism is also bijective, we say it is an **isomorphism**.

Definition. If $h : \mathcal{A} \to \mathcal{B}$ be a homomorphism, then we define the **kernel** of h, written $\ker(h)$, by

$$\ker(h) = \{(a_1, a_2) \in A \times A : h(a_1) = h(a_2)\}$$

Definition. If $\mathcal{A}$ is an algebra of type $\mathcal{F}$ and θ is an equivalence relation on A that further satisfies that for each $f \in \mathcal{F}$ and elements $a_i, b_i \in A$, if $a_i \theta b_i$ for all $1 \leq i \leq n$ then

$$f(a_1, ..., a_n) \theta f^(b_1, ..., b_n)$$

then we say θ is a **congruence** on $\mathcal{A}$.

Definition. Let $\mathcal{A}$ be an algebra and θ a congruence on $\mathcal{A}$. Then the **quotient algebra** $\mathcal{A}/\theta$ of $\mathcal{A}$ by θ is the algebra with universe A/θ such that for all $f \in \mathcal{F}$ and $a_1, ...a_n \in A$,

$$f^{\mathcal{A}/\theta}(a_1/\theta, ..., a_n/\theta) = f^{\mathcal{A}}(a_1, ..., a_n)/\theta$$

All of the above definitions I have taken nearly verbatim from [BurrSank], in which the authors make the point that nearly all of the algebras which have been studied up to this point have families of operations with maximum arity of two. In the pages that follow, I will introduce a new type of algebra that has an operator of arity three, and one that, it will be seen, helps to generalize certain structures of binary first-order logic.

1 Definition, Examples

Definition. A **logic** is an algebra $\mathcal{L} = (L, \circ, \sigma)$, with a ternary operation $\circ$ and a unary operation σ such that:

L1 σ is a bijection

L2 for all $a, b, c \in L$:

(i) $\circ(a, b, c) = \circ(a, c, b) = a$ if b or $c = a$

(ii) $\circ(a, b, c) = \circ(a, c, b) = \sigma(a)$ otherwise

Some terminology:

1. L is the **truth set**, whose elements are **truth values**. The **order** of a logic is the cardinality of L.
2. σ is the **negation permutation**.
3. $\circ$ is the **operator function** which assigns an **assimilating operator** to every truth value. For notational convenience, we shall often write $\circ(a, b, c)$ as $b \circ_a c$.
4. Furthermore, if we have a finite truth set, we can enumerate the truth values by $\{t_1, ..., t_n\}$ and express the negation permutation as an element of S_n. Thus we will often adopt the convention when dealing with finite truth sets of writing $\mathcal{L}_\sigma = (\{t_1, ...t_n\}, \circ, \sigma)$. The caveat of this choice is that one must be careful not to omit fixed points, since the recorded negation permutations will also key a reader in to the order of $\mathcal{L}$.

Example 1.
$\mathcal{L}_{(12)} = (\{T, F\}, \circ, \neg)$. Suppose P is some object capable of "having" truth value. Then,

P	¬(P)
T	F
F	T

Then the $\circ$ function will yield the binary operators $\circ_T$ and $\circ_F$ which will behave as follows:

P	Q	P$\circ_T$Q
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	P$\circ_F$Q
T	T	T
T	F	F
F	T	F
F	F	F

A notational switch from $\circ_T$ to $\vee$ and $\circ_F$ to $\wedge$ reveals the familiar truth tables of first-order binary logic.

Definition. A **weak logic** is an algebra $(L = L_D \cup L_I, \circ, \sigma)$ where

WL1 $L_D \cap L_I = \emptyset$

WL2 $\circ : L_D \times L^2 \to L$

and both L1, and L2(i) hold.

1. We call the elements of L_D the **dominant** truth values and the elements of L_I the **intermediate** truth values.

2. Any logic that is not a weak logic is a **strong logic**.

This definition allows us to pull many existing multi-valued logics into our framework.

Example 2 (The Kleene Logic).
$\mathcal{L}_{(12)(3)} = (\{T, F\} \cup \{N\}, \sigma, \circ)$ where,

x	σ(x)
T	F
N	N
F	T

so,

$\circ_T$	**T**	**N**	**F**
T	T	T	T
N	T	N	N
F	T	N	F

$\circ_F$	**T**	**N**	**F**
T	T	N	F
N	N	N	F
F	F	F	F

Although our first two examples are familiar, there are many examples of logics which have not already been seriously studied, many of which will be important to our discussion later on. In particular, finite logics with monocyclic negation permutations, which we shall call **cyclic logics (Definition)** will be important in generalizing the ideas of natural deduction and propositional logic. The most basic example of a finite cyclic logic of higher order than the classical binary is the following,

Example 3.
$\mathcal{L}_{(123)} = (\{a, b, c\}, \circ, \sigma)$:

x	σ(x)
a	b
b	c
c	a

$\circ_a$	**a**	**b**	**c**
a	a	a	a
b	a	b	b
c	a	b	b

$\circ_b$	**a**	**b**	**c**
a	c	b	c
b	b	b	b
c	c	b	c

$\circ_c$	**a**	**b**	**c**
a	a	a	c
b	a	a	c
c	c	c	c

Note that we could also have a logic with the same truth set, but a different negation permutation that would behave entirely differently. For example,

Example 4.
$\mathcal{L}_{(1)(23)} = (\{a, b, c\}, \sigma, \circ)$:

x	σ(x)
a	a
b	c
c	b

$\circ_a$	**a**	**b**	**c**
a	a	a	a
b	a	a	a
c	a	a	a

$\circ_b$	**a**	**b**	**c**
a	c	b	c
b	b	b	b
c	c	b	c

$\circ_c$	**a**	**b**	**c**
a	b	b	c
b	b	b	c
c	c	c	c

Lastly, although they will not feature heavily in this paper, perhaps it is worth providing an example of a logic with a truth set that is not finite.

Example 5.
$\mathcal{L}_\sigma = (\mathbb{Z}, \sigma, \circ)$: for all $n \in \mathbb{Z}, \sigma(n) = n + 1$.

2 Logic Homomorphisms, Theorems.

Definition. Let $\mathcal{L}_\sigma$ and $\mathcal{L}_\tau$ be two logics.

1. $h : \mathcal{L}_\sigma \to \mathcal{L}_\tau$ is a **logic homomorphism** if and only if for all $x, y, z \in L_\sigma$,
   (i) $h(\sigma(x)) = \tau(h(x))$
   (ii) $h(\circ_1(x, y, z)) = \circ_2(h(x), h(y), h(z))$.
2. $f : \mathcal{L}_1 \to \mathcal{L}_2$ is a **logic isomorphism** if and only if it is a bijective logic homomorphism.

Cycle-Type Theorem. Two logics are isomorphic if and only if their negation permutations have the same cycle type.

Proof: Suppose $\alpha : \mathcal{L}_\sigma \to \mathcal{L}_\tau$ and α isomorphism. Then for all $x \in L_\sigma$, since $\alpha(\sigma(x)) = \tau(\alpha(x))$, $|x| = |\alpha(x)|$, so σ, τ have the same cycle type. Suppose σ, τ have the same cycle type. For each cycle σ_i in σ, choose x not fixed by σ_i and define $\alpha : \mathcal{L}_\sigma \to \mathcal{L}_\tau$ such that $\alpha(x)$ is not fixed by τ_j where $|\tau_j| = |\sigma_i|$ and $\alpha(\sigma(x)) = \tau(\alpha(x))$. Clearly then α is a surjective homomorphism. Suppose $\alpha(x_1) = \alpha(x_2)$, then it must be that x_1 and x_2 belong to the same permutation orbit, thus $x_2 = \sigma^m(x_1)$ for some m. But then

$$\alpha(x_1) = \alpha(x_2) = \alpha(\sigma^m(x_1)) = \tau^m(\alpha(x_1))$$

and thus $x_2 = \sigma^m(x_1) = x_1$, so α is injective and thus an isomorphism. $\square$

Homomorphism Characterization Theorem. Suppose $h : \mathcal{L}_\sigma \to \mathcal{L}_\tau$ is a logic homomorphism with $\sigma = \sigma_1\sigma_2...\sigma_m$, and $\tau = \tau_1\tau_2...\tau_k$, where each σ_i, τ_j are disjoint cycles.

(1) If for some $x \in L_\sigma$, $h(x)$ not fixed by τ_j, then for all $y \in L_\tau$ not fixed by τ_j, there exists $x \in L_\sigma : h(x) = y$. We call this property **intracycle surjective (Definition)**.

(2) If for some $x \in L_\sigma$, $h(x)$ not fixed by τ_j, then for all n, $h(\sigma^n(x))$ is not fixed by τ_j. We call this property **cycle-wise well-defined (Definition)**.

(3) Either,

  (a) There are m disjoint cycles in $h(\sigma)$, or

  (b) there exists $\sigma_i, \sigma_j : h(\sigma_i) = h(\sigma_j)$ and $|h(\sigma_i)| = |h(\sigma_j)| = 1$.

Proof: (1) Suppose there exists $y \in L_\tau$ not fixed by some τ_j. If there exists $x \in L_\sigma$ such that $h(x)$ not fixed by τ_j, then there exists $m : y = \tau^m(h(x)) = h(\sigma^m(x))$, since h is a homomorphism, so $h : \sigma^m(x) \to y$ and is intracycle surjective. (2) Follows from the properties of logic homomorphisms. (3) It follows from (2) that there cannot be more m disjoint cycles in $h(\sigma)$. Assume there are fewer, i.e. there exists $\sigma_a, \sigma_b : h(\sigma_a) = h(\sigma_b)$. By (1) and (2), it is possible to choose $x, y, z \in L_\sigma$ where x not fixed by σ_a, y, z not fixed by σ_b, and $h(x) = h(y)$. Since $\circ(x, y, z) = \sigma(x)$, $h(\circ(x, y, z)) = h(\sigma(x)) = \tau(h(x))$. But also, $h(\circ(x, y, z)) = \circ(h(x), h(y), h(z)) = h(x)$ since $h(x) = h(y)$, and so $h(x) = \tau(h(x))$ and thus $|h(\sigma_a)| = |h(\sigma_b)| = 1$. $\square$

Let us now apply several of the definitions from Universal Algebra to our setting.

Definition. A **congruence** on a logic $\mathcal{L}_\sigma$ is an equivalence relation θ on $L \times L$ that further satisfies

(i) if $x \; \theta \; y$, then $\sigma(x) \; \theta \; \sigma(y)$

(ii) if $x_1, y_1, z_1 \; \theta \; x_2, y_2, z_2$ respectively, then $\circ(x_1, y_1, z_1) \; \theta \; \circ(x_2, y_2, z_2)$

Definition. Let θ be a congruence on a logic $\mathcal{L}_\sigma$. Then the **quotient logic** $\mathcal{L}_\sigma/\theta = (L/\theta, \circ, \tau)$ such that for all $x, y, z \in L$,

$$\tau(x/\theta) = \sigma(x)/\theta$$

$$\circ(x/\theta, y/\theta, z/\theta) = \circ(x, y, z)/\theta$$

It is elementary to check that any such quotient logic indeed satisfies the logic axioms.

Definition. If $h : \mathcal{L}_\sigma \to \mathcal{L}_\tau$ is a homomorphism, its **kernel** is $\{(x, y) \in L_\sigma^2 : h(x) = h(y)\}$.

Lemma. The kernel of a logic homomorphism is a congruence.

Proof: Let $h : \mathcal{L}_\sigma \to \mathcal{L}_\tau$ a homomorphism. Clearly the kernel is an equivalence relationship. Furthermore, suppose $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \ker(h)$. Then $\tau(h(x_1)) = \tau(h(x_2))$ but since h is a homomorphism, $h(\sigma(x_1)) = h(\sigma(x_2))$ so $(\sigma(x_1), \sigma(x_2)) \in \ker(h)$. A similar argument shows $(\circ(x_1, y_1, z_1), \circ(x_2, y_2, z_2)) \in \ker(h)$. $\square$

We are now equipped to prove the First Logic Isomorphism Theorem. Note that since logics are algebras, all results of Universal Algebra should hold, including all isomorphism theorems, and thus one might invoke this fact and forgo proof. That shall be exactly our strategy for the Second and Third Logic Isomorphism Theorems, but it is the author's opinion that at least one independent proof of an algebraic result in the setting of logics will be an illuminating exercise.

First Logic Isomorphism Theorem. Let $h : \mathcal{L}_\sigma \to \mathcal{L}_\tau$ be a logic homomorphism, and π the natural homomorphism $\pi : \mathcal{L}_\sigma \to \mathcal{L}_\sigma/\ker(h)$. Then there exists $\alpha : \mathcal{L}_\sigma/\ker(h) \to \mathcal{L}_\tau$ defined by $\alpha(x/\ker(h)) = h(x)$ that is an isomorphism.

Proof: For convenience, let us relabel $\ker(h) = \kappa$. Since π is a homomorphism, $\alpha(\sigma(x/\kappa)) = \alpha(\sigma(x)/\kappa) = h(\sigma(x))$. Since h is a homomorphism, $h(\sigma(x)) = \sigma(h(x)) = \sigma(\alpha(x/\kappa))$. A similar argument shows $\alpha(\circ(x/\kappa, y/\kappa, z/\kappa)) = \circ(\alpha(x/\kappa), \alpha(y/\kappa), \alpha(z/\kappa)))$, so α is an homomorphism. For any $h(x) \in L_\tau$, $\alpha(x/\kappa) = h(x)$, so α is surjective. If $h(x) = h(y)$ then by the definition of the kernel $x/\kappa = y/\kappa$. So α is injective. $\square$

Second Logic Isomorphism Theorem. If ϕ, θ are congruences on a logic $\mathcal{L}$, such that $\theta \subseteq \phi$, then the map

$$\alpha : \frac{L/\theta}{\phi/\theta} \to L/\phi$$

defined by

$$\alpha\left(\frac{x/\theta}{\phi/\theta}\right) = x/\phi$$

is an isomorphism from $\frac{\mathcal{L}/\theta}{\phi/\theta}$ to $\mathcal{L}/\phi$.

Proof: $\mathcal{L}$ is an algebra. $\square$

Definition. A **sublogic** of a logic $\mathcal{L}$ is any subset of its truth set which itself forms a logic.

Third Logic Isomorphism Theorem. If $\mathcal{K}$ is a sublogic of $\mathcal{L}$ and θ is a congruence on $\mathcal{L}$ then

$$\mathcal{K}/\theta \restriction_K \cong \mathcal{K}^\theta/\theta \restriction_{K^\theta}$$

Where $\mathcal{K}^\theta$ is the sublogic generated by $K^\theta = \{x \in L : K \cap x/\theta \neq \emptyset\}$.

Proof: $\mathcal{L}$ is an algebra. □

3 Logic Arithmetic, The Category of Logics

Suppose we want to delve more into the question of how to build new logics from old ones. It may have been noticed, for example, that we have yet to touch on the concept of direct product, one that seems fundamental to any algebraic structure. Thus far, we have successfully commandeered many concepts form Universal Algebra in helping us to understanding this new and interesting mathematical structure, and certainly there would be nothing to prevent us from continuing through the many results of UA (of which a generalization of direct product is but one) and adapting them to our particular setting. It is my hope, however, that at this point we are already developing some flavor for what this process would feel like, and so for the sake of keeping things interesting, let us take a quick detour from Universal Algebra into the world of Category Theory, an even more abstract framework through which to view combinations of mathematical structures. I will begin by presenting some fundamental notions from the discipline, then show how we can use these to help us better understand the world of logics, and offer an alternative lens through which to view the concepts of sum and product.

Definition. A **category** consists of the following pieces:

1. **objects**, such as $A, B, C, ...$
2. **maps** between objects, $A \xrightarrow{f} B, B \xrightarrow{g} C, ...$
3. **identity maps** (one per object), $A \xrightarrow{1_A} A$ ...
4. **compositions of maps** which assign to each pair of maps another map, i.e. if $A \xrightarrow{f} B \xrightarrow{g} C$ then we have $A \xrightarrow{g \circ f} C$

that obey the following laws

C1 : $f \circ 1_A = f$ and $1_A \circ f = f$

C2 : $h \circ (g \circ f) = (h \circ g) \circ f$.

Example 6. **Set** is the category of sets (which are its objects), with set containments as its maps.

Example 7. **Grp** is the category of groups, with group homomorphisms as its maps.

Definition. In a category,

(a) The **product** of two objects A and B is another object P such that

   (i) there exist maps from P into both A and B

   (ii) and if there is another object X with maps into both A and B then there exists a unique map from X into P such that the composition of this map with the maps from P into A and B yield the maps from X into A and B.

(b) The **coproduct** is the same definition with the maps reversed, i.e. another object C such that

   (i) there exist maps from both A and B into C

   (ii) and if there is another object Y with maps into it from both A and B then there exists a unique map from C into Y such that the composition of the maps from A and B into C with this map yield the maps from A and B into Y.

Example 8. In **Set** and **Grp**, the product and coproduct of any two objects is just their direct product and direct sum, respectively.

Definition. In a category,

(a) An **initial object** I is an object for which

   (i) there exists a map into any other object,

   (ii) and if there exists another such object X with maps into every object, then there exists a unique map from X into I, such that composing this map with the map from I into a given object will yield the map from X into the object.

(b) A **final object** is the same thing but with the maps reversed, i.e. an object O for which

   (i) there exists maps from any object into it,

   (ii) and if there exists another such object Y that can be mapped to by every object then there must exist a unique map from O into Y, such that the composition of the map from a given object into O with this map yields the map from the given object into Y.

Example 9. In **Set**, the initial object is just the empty set, and all singleton sets are final objects. In **Grp**, the trivial group is both the initial and final object.

This is as far into category theory as we need to venture for our purposes. We are now ready to apply these ideas to the world of logics.

Definition. The set of all logics form a category which we will call **Log**. Its objects are logics, and its maps are logic homomorphisms.

Definition. Let $\mathcal{L}_\sigma, \mathcal{L}_\tau$ be two logics. We define,

1. their **sum** $\mathcal{L}_\sigma + \mathcal{L}_\tau = (L_\sigma \uplus L_\tau, \mu, \circ) : \mu(x) = \begin{cases} \sigma(x) & \text{if } x \in L_\sigma \\ \tau(x) & \text{if } x \in L_\tau \end{cases}$

2. their **product** $\mathcal{L}_\sigma \times \mathcal{L}_\tau = (L_\sigma \times L_\tau, \mu, \circ) : \mu((x, y)) = (\sigma(x), \tau(y))$

Proposition. Logic multiplication and logic addition, as defined above, give the categorical product and coproduct of two logics, respectively.

Proof: Let $\mathcal{L}_\sigma$ and $\mathcal{L}_\tau$ be two logics. Clearly there are two projection homomorphisms π_σ, π_τ from $\mathcal{L}_\sigma \times \mathcal{L}_\tau$ to $\mathcal{L}_\sigma$, $\mathcal{L}_\tau$ respectively defined by $\pi_\sigma((s,t)) = s$, $\pi_\tau((s,t)) = t$. Suppose now there exists another logic $\mathcal{L}_\mu$ that has homomorphic mappings into $\mathcal{L}_\sigma$ and $\mathcal{L}_\tau$ via the homomorphisms f and g, respectively. Let α be the homomorphism from $\mathcal{L}_\mu$ into $\mathcal{L}_\sigma \times \mathcal{L}_\tau$ defined as follows,

$$\alpha(x) = (f(x), g(x))$$

then it is clear that $\pi_\sigma \circ \alpha = f$, $\pi_\tau \circ \alpha = g$, so $\mathcal{L}_\sigma \times \mathcal{L}_\tau$ is indeed the product of $\mathcal{L}_\sigma$ and $\mathcal{L}_\tau$.

Now let us examine the embedding maps η_σ, η_τ from $\mathcal{L}_\sigma$ and $\mathcal{L}_\tau$ into $\mathcal{L}_\sigma + \mathcal{L}_\tau$, defined by $\eta_\sigma(s) = s$ and $\eta_\tau(t) = t$. Suppose there were to exist another logic $\mathcal{L}_\nu$ mapped into by both $\mathcal{L}_\sigma$ and $\mathcal{L}_\tau$ via the homomorphisms j and k respectively. Let β be the homomorphism from $\mathcal{L}\sigma + \mathcal{L}_\tau$ into $\mathcal{L}_\nu$ defined by

$$\beta(x) = \begin{cases} j(x) & \text{if } x \in L_\sigma \\ k(x) & \text{if } x \in L_\tau \end{cases}$$

then it is clear that $\beta \circ \eta_\sigma = j$, $\beta \circ \eta_\tau = k$, so $\mathcal{L}_\sigma + \mathcal{L}_\tau$ is indeed the coproduct of $\mathcal{L}_\sigma$ and $\mathcal{L}_\tau$. $\square$

Logic Decomposition Theorem. All logics can be decomposed as the sum of cyclic logics.

Proof: Since all permutations can be written as the product of disjoint cycles, for a given logic $\mathcal{L}_\sigma$, where $\sigma = \sigma_1\sigma_2...\sigma_n$ all disjoint, it follows trivially that $\mathcal{L}_\sigma = \mathcal{L}_{\sigma_1} + \mathcal{L}_{\sigma_2} + ... + \mathcal{L}_{\sigma_n}$ $\square$

Definition. The **null-logic** $\mathcal{L}_\emptyset$ is the 0-logic that has as its truth set the empty set. It has no negation permutation nor operator function.

Definition. A **unit-logic** $\mathcal{L}_1$ is any 1-logic that has a singleton truth set.

Proposition. The null-logic is the initial object in **Log**.

Proof: The null-logic maps trivially into every other logic. Suppose we have a non-empty logic $\mathcal{L}_\sigma$ where $\sigma = \sigma_1\sigma_2...\sigma_n$ all disjoint. Let $\sigma' = \sigma_1'\sigma_2...\sigma_n$ where the cycle length of σ_1' is one longer than σ_1. Then via the Homomorphism Characterization Theorem (2) we know that there can exist no homomorphism from $\mathcal{L}_\sigma$ to $\mathcal{L}_{\sigma'}$. Thus the null-logic is the only object that maps into every other object, so any object that maps into every object must trivially map into the null logic. $\square$

Proposition. Any unit logic or sum of unit logics is a final object in **Log**.

Proof: Any logic can map trivially into a unit logic via the homomorphism that sends every element of the domain to the single element in the codomain. Suppose we have a logic $\mathcal{L}_\sigma$ where σ contains at least one cycle of length greater than one. Then by the reasoning in the previous proof, we know, for example, that there does not exist a homomorphism from any unit logic to $\mathcal{L}_\sigma$, so it is only to unit logics and sums of unit logics that any logic can be mapped. But note that sums of unit logics fail to satisfy the uniqueness criterion in part (i) of the definition. $\square$

4 Ramifications to Other Areas of Mathematics

Given our particular generalization of the structure of logic, it is not un-interesting stroll through various mathematical topics, and wherever we encounter a binary, replace it with an arbitrary number. The concept of ordering, for example, can be generalized in any metric space with the following definition,

Definition. If (X, δ) is a metric space and $A, B, \alpha, \beta \in X$, we say $\alpha <_A^B \beta$ or equivalently $\beta <_B^A \alpha$ if and only if

$$\frac{\delta(\alpha, A)}{\delta(\alpha, B)} <^* \frac{\delta(\beta, A)}{\delta(\beta, B)}$$

and call X the **ordering space** and A, B the **poles** of the ordering.

We can thus define pairwise orderings on any number of poles. To help picture what a three-valued ordering might look like, think of a triangle (closed or open) in $\mathbb{R}^2$ with vertices A, B and C. Points in this triangle would be ordered with respect to any two given poles by their distance from the two poles as given above. In turn, we can utilize this generalization to construct,

Definition. Suppose X is an ordering space with n poles P_n. A relation $\leq$ on $(X \times X)$ that satisfies for all $\alpha, \beta, \gamma, P_i, P_j \in$ X

(1) $\alpha \leq_{P_j}^{P_i} \alpha$

(2) If $\alpha \leq_{P_j}^{P_i} \beta$ and $\beta \leq_{P_j}^{P_i} \gamma$ then $\alpha \leq_{P_j}^{P_i} \gamma$

we shall call an **n-valued partial ordering** on X, and a set with such an ordering is an **Po(n)set**. In particular we call it a **weak** Po(n)set. A Po(n)set that further satisfies,

(3) If $\alpha \leq_{P_k}^{P_l} \beta$ and $\beta \leq_{P_k}^{P_l} \alpha$ for all k, l, then $\alpha = \beta$

is called a **strong** Po(n)set.

Example 10.

(a) Take X to be $\mathbb{R}^2$ with the usual distance function and $A, B \subset \mathbb{R}^2, A \cap B = \emptyset$. Then X is a weak Po2set.

(b) Take X to be $\mathbb{R}^2$ with the usual distance function and $A, B, C \subset \mathbb{R}^2$, all disjoint. Then X is a strong Po3set.

Now, these two examples happen also to be **totally ordered** Po(n)sets, or **n-chains**, i.e. they further satisfy,

(4) $\alpha \leq_{P_j}^{P_i} \beta$ or $\beta \leq_{P_j}^{P_i} \alpha$

so they are easy to visualize. Note that 1 dimension is enough to visualize a 2-chain, but when we try to visualize an arbitrary Po2set (for the sake of lattice theory, for example), we must utilize two dimensions. Thus, unless one is terribly comfortable in four dimensions, visualizing an arbitrary Po3set might be difficult.

If, on top of what we have so far, we further define,

Definition. If we have n poles P in a metric space X with S $\subset$ X, then $^{\mathbf{P}_i}$**cap(S)** $=$ α such that for all $p \in$ S, and for for all $j \neq i, \alpha \leq_{P_i}^{P_j} p$ and if $q \leq_{P_i}^{P_j}$ S, then $q \leq_{P_i}^{P_j} \alpha$,

which is just a generalization of the concept of infimum and supremum, we can go on to discuss,

Definition. Suppose L is a Po(n)set. If, for all $\alpha, \beta \in$ L, there exists a ^{P_i}cap($\{\alpha, \beta\}$) $\in$ X, for every pole P$_i$, then we call L an **n-lattice**.

Notice that just as in a conventional lattice $\inf(x, y) = x \wedge y$, in general $^{\mathrm{A}}\mathrm{cap}(x, y) = x \circ_{\mathrm{A}} y$, as defined earlier in this paper. For the sake of brevity, we will end this particular exploration here, but point out that it might be of interest to study these multi-valued lattices, if not just for their own sake, perhaps in conjunction with probability theory, and thus construct a theory of multi-valued probability consistent with the notions of this paper. Furthermore, it should be possible to generalize in a similar manner the concepts of union and intersection, i.e.

Definition. if $\mathcal{L}_\sigma$ is a logic with $A \in L$, and P, Q are sets,

$$P \bigcirc_A Q = \{x : x \in P \circ_A x \in Q\}$$

and thus discover a particular generalization of Set Theory or Topology.

5 Formal Languages for Finite Cyclic Logics

Formation Rules
To build formal languages around multi-valued logics, one has to modify only slightly the formation rules of those built around binary logic. We might attempt to do so, for example, with the formation rules of SL, and construct a formal language SL_σ built around an arbitrary logic $\mathcal{L}_\sigma$:

1. Each sentence letter $\in \{P, Q, R, P_i, Q_j, R_k : \text{for all } i, j, k \in \mathbb{N}\}$ is a formula.
2. If ϕ is a formula, so is $\sigma^m(\phi)$, for all $m \in \mathbb{N}$.
3. If ϕ and ψ are formulae, so is $\phi \circ_x \psi$ for all $x \in L$.
4. Nothing else is a formula.

Definition. In deference to conventional logical notation, we take $|\phi|_I$ to mean the **truth-value of ϕ under** a given interpretation **I**.

In such a language, (just as in SL), any combination of formulae over a single binary operation is associative and so we will sometimes adopt the following notation to deal with long strings of formulae connected by the same binary operator:

$$\circ_x(P_1, P_2, ..., P_n) = P_1 \circ_x P_2 \circ_x ... \circ_x P_n$$

Generalized DeMorgan's Law If P and Q are two propositions in a cyclic logic $\mathcal{L}$, with $x, y \in L$, then, if $\sigma^m(x) = y$,

$$P \circ_x Q = \sigma^{n-m}(\sigma^m(P) \circ_y \sigma^m(Q))$$

Proof: In any given interpretation I, by definition $|P \circ_x Q|_I = x$ or $|P \circ_x Q|_I = \sigma(x)$. If $|P \circ_x Q|_I = x$, then $|P|_I = x$ or $|Q|_I = x$. Either way, at least one of $|\sigma^m(P)|_I, |\sigma^m(Q)|_I = y$, and so $|\sigma^m(P) \circ_y \sigma^m(Q)|_I = y$ and thus $|\sigma^{n-m}(\sigma^m(P) \circ_y \sigma^m(Q))|_I = x$. On the other hand, if $|P \circ_x Q|_I = \sigma(x)$, neither $|P|_I = x$ nor $|Q|_I = x$. So $|\sigma^m(P)|_I, |\sigma^m(Q)|_I \neq y$, and so by definition $|\sigma^m(P) \circ_y \sigma^m(Q)|_I = \sigma(y)$ and thus $|\sigma^{n-m}(\sigma^m(P) \circ_y \sigma^m(Q))|_I = \sigma(x)$. □

Here, for the first time, it become evident why we are restricting out attention to cyclic logics, since in a non-cyclic logic there does not necessarily exist an $m \in \mathbb{N} : \sigma^m(x) = y$.

Functional Completeness. Let $\mathcal{L}$ be a logic and $*$ be a binary truth functor with outputs specified by an arbitrary truth table. If $*$ is commutative, and the truth-value outputs of $*$ are all of the form x and $\sigma(x)$ for some $x \in L$, then $*$ may be written in terms of binary operators given by the operator function $\circ$.

Proof: Suppose P and Q are two formulae in SL_σ conjoined by an arbitrary truth functor $*$ with outputs of the form x and $\sigma(x)$ for some $x \in L$. Observe the following method.

1. Identify the $y \in L : \sigma(y) = x$

2. For every line of the truth table for $*$ (i.e. every interpretation I) that yields the output x construct a $\phi_i = \circ_y(\sigma^{j_1}(P), ..., \sigma^{j_{n-1}}(P), \sigma^{k_1}(Q), ..., \sigma^{k_{n-1}}(Q))$ where the $\sigma^{j_m}(P), \sigma^{k_p}$ are all possible distinct permutations of P and Q *excluding* those for which $|\sigma^{j_m}(P)|_I, |\sigma^{k_p}(Q)|_I = y$.

3. The formula $\circ_x(\phi_1, \phi_2, ..., \phi_q)$ will have a truth table identical to $*$. □

Example 11. Suppose $\mathcal{L}$ is a cyclic 3-logic with

x	σ(x)
a	b
b	c
c	a

$*$	**a**	**b**	**c**
a	a	b	a
b	b	b	b
c	a	b	b

Since the outputs of $*$ are of the form, a and $\sigma(a) = b$ we can re-write $*$ in terms of the operator function $\circ$ according to the procedure given above.

1. We identify c as the element in L such that $\sigma(c) = a$.

2.
$$\phi_1(\text{from line 1}) = \circ_c(P, \sigma(P), Q, \sigma(Q))$$
$$\phi_2(\text{from line 3}) = \circ_c(P, \sigma(P), \sigma(Q), \sigma^2(Q))$$
$$\phi_3(\text{from line 7}) = \circ_c(\sigma(P), \sigma^2(P), Q, \sigma(Q))$$

3.

P	Q	$\circ_a(\phi_1, \phi_2, \phi_3) =$ $\circ_a(\circ_c(P, \sigma(P), Q, \sigma(Q)), \circ_c(P, \sigma(P), \sigma(Q), \sigma^2(Q)), \circ_c(\sigma(P), \sigma^2(P), Q, \sigma(Q)))$
a	a	a
a	b	b
a	c	a
b	a	b
b	b	b
b	c	b
c	a	a
c	b	b
c	c	b

With this result in hand on could, if one so desired, define "conditional" truth-functors in a given logic analogous to $\rightarrow$ or $\leftrightarrow$. Suppose we characterize the behavior of $\rightarrow$ as follows,

$$|P \rightarrow Q|_I = F \text{ when } |P|_I = T \text{ and } |Q|_I = F, \text{ and } |P \rightarrow Q|_I = T \text{ otherwise.}$$

A natural generalization for a conditional truth functor $\rightarrow_x$ for propositions in a given $\mathrm{S}_n\mathrm{L}$ might be

$$|P \rightarrow_x Q|_I = \sigma(x) \text{ when } |P|_I = x \text{ and } |Q|_I = \sigma(x), \text{ and } |P \rightarrow_x Q|_I = x \text{ otherwise.}$$

Similarly, we might describe the behavior of $\leftrightarrow$ in binary logic as either

1. $|P \leftrightarrow Q|_I = T$ when $|P|_I = |Q|_I$, and $|P \leftrightarrow Q|_I = F$ otherwise.

   or perhaps we prefer

2. $|P \leftrightarrow Q|_I = |(P \to Q) \wedge (Q \to P)|_I$.

But now we have encountered an interesting subtlety. In the binary logic we are used to, the above definitions are equivalent (ie. the truth functors so described have equivalent truth-tables). However, a quick foray into high-ordered logics will reveal this is not necessarily the case. In the 3-logic given in Example 2, if we generalize the above characterizations to

1. $|P \leftrightarrow_x Q|_I = x$ when $|P|_I = |Q|_I$, and $|P \leftrightarrow_x Q|_I = \sigma(x)$ otherwise.

   and

2. $|P \leftrightarrow_x Q|_I = |(P \to Q) \circ_{\sigma(x)} (Q \to P)|_I$.

then the truth table for 1 is given by,

$\leftrightarrow_x$	**a**	**b**	**c**
a	a	b	b
b	b	a	b
c	b	b	a

and the truth table for 2 by,

$\leftrightarrow_x$	**a**	**b**	**c**
a	c	b	c
b	b	c	c
c	c	c	c

which are certainly not equivalent. Clearly one must tread carefully when generalizing results from binary logic as there are many redundancies that untangle themselves only in higher-order logics. If anything, this should be an encouragement to the reader, and perhaps the first hint of a justification for why such logics would be worthy of further study.

The implications of this particular type of multi-valued logic are hardly limited to propositional languages, however. Note that, in a given universe of discourse $\mathcal{U}$,

$$\forall x = \bigwedge_{\mathcal{U}} x$$

$$\exists x = \bigvee_{\mathcal{U}} x$$

and so it would be possible to generalize the concept of quantification as follows:

$$\ominus_A x = \circ_A(\{x \in \mathcal{U}\})$$

yet once again we shall leave the fleshing out of the details of this generalization for another paper.

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