An Alternate Proof for the Jones Polynomial of T(2,n) Torus Knots Using Trip Matrices

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1 Introduction

The Jones polynomial is an important knot invariant introduced in 1984 by Vaughn Jones. Up to this point, the Jones polynomial has been able to detect triviality and it is one of the few invariants that has deep connections to quantum mechanics. There are multiple ways to compute the Jones polynomial and in this paper we will explore a method introduced by Zulli [Zulli, 1995] using what he calls trip matrices. In particular, we will focus on using trip matrices to compute the Jones polynomial of T(2,n) torus knots. These knots are an important class of knots that are characterized by their ability to be drawn on a torus. Jones proved an explicit formula for the Jones Polynomial of all torus knots, but the proof relies on heavy machinery from Abstract Algebra. We provide a more elementary proof of this formula for T(2, n) knots using these trip matrices and basic Linear Algebra.

2 Background

2.1 Basics of Knot Theory

We provide a basic introduction to the relevant parts of knot theory for the paper. For more information or details on the topic, see for example [Adams, 2004].

A **knot** is a closed loop in $\mathbb{R}^3$, having no thickness that does not intersect itself. A fundamental question in knot theory is: Can you manipulate one knot in $\mathbb{R}^3$ without cutting it to look the same as another knot in $\mathbb{R}^3$? This type of manipulation is called an **ambient isotopy**. We say that two knots are **equivalent** if there exists an ambient isotopy from one knot to another knot. Thus, this fundamental question rephrased is: Are two given knots equivalent?

To visualize knots in $\mathbb{R}^2$, we draw a projection of the knot where the type of crossing is indicated with a break in the string. See figure 3 for some examples. On a projection of a knot, if the strand underneath goes from right to left the crossing is defined as positive (Figure 1) and if it goes from left to right it is defined as negative (Figure 2).

Figure 1:

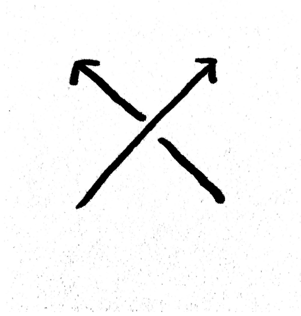
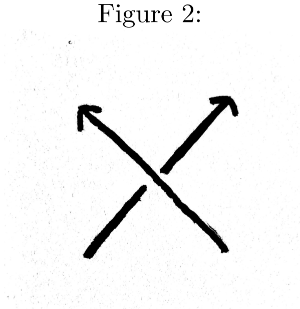
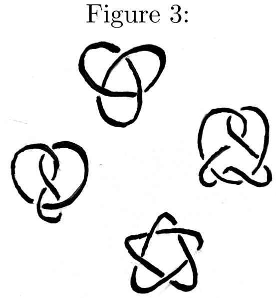


Figure 2:



We would like to be able to determine if two knots are equivalent by showing equivalence of the projections. Reidemeister established this fact and showed you only need three moves for equivalence.

Figure 3:

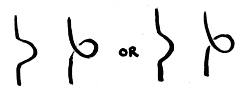


Reidemeister's Theorem states that if we have two distinct projections of the same knot, we can get from one projection to the other by a series of

Reidemeister moves and planar isotopies [Reidemeister, 1972].

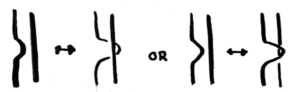
There are three Riedemeister moves. The first Riedemeister move is to twist or untwist a strand in the knot, see figure 4.

Figure 4:



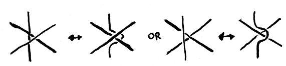
The second is to add two crossings or remove two crossings, see figure 5.

Figure 5:



The third Riedemeister move is defined as moving a strand from one side of a crossing to the other, see figure 6.

Figure 6:



Because there is an infinite sequence of Riedemeister moves, you could try to manipulate a given projection of a knot. The question of which knots are equivalent is difficult, so we introduce the concept of an **invariant**.

An **invariant** is a property of a knot the remains unchanged through different projections of the same knot. Thus, in terms of diagrams, an invariant must remain unchanged by the three Reidemeister moves. The first polynomial knot invariant was introduced by Alexander in 1923. Several years later, Jones introduced a different polynomial knot invariant known as the Jones polynomial. We will focus on computing this for a special class of knots, torus knots.

Torus knots are knots that lie on an unknotted torus.

Definition 1. *A $T(m,n)$ torus knot wraps n times around a circle inside the torus and m times around a line through the hole in the torus.*

Torus knots, unlike other categories of knots, often have very nice properties and some invariants can be easily computed. For example, below are some of the properties:

- $T(m,n)$ is equivalent to $T(n,m)$ for all n,m.
- Torus knots are chiral. That is, their mirror images are not equivalent. Hence, $T(m,n)$ denotes two different knots.
- The crossing number is known for all torus knots. In fact, c(T(m,n))=min{m(n-1),n(m-1)}.
- The unknotting number for torus knots is also known which is 1/2(m-1)(n-1) [Murasagi, 1991].

2.2 Braids and Their Connection to Knots

Definition 2. *An n-braid is a set of n-strands attached to a horizontal bar at the top and at the bottom with the additional requirement that at any given height along the braid, each strand only attains that height once.*

See below for an example of a 2-braid.

Figure 7:

Braids and knots are very closely related since, given any braid, we may close it to obtain a knot or link (where a link is simply several knots that are linked together in some fashion). To close the braid, simply pull the bottom bar around and glue it to the top bar. Somewhat surprisingly, the reverse is always true which was proved by Alexander.

Theorem 1. *Every knot or link can be represented as a closed braid [Alexander, 1923].*

Due to Alexander's Theorem, any statement about a knot can be transformed into a statement about braids (or vice versa). The torus knots we will study here have very nice closed braid representations. Before stating the propositions, we need some notation for the crossings in a braid. Each crossing, as we look at the braid, from top to bottom will be denoted with a σ_i or σ_i^{-1}, where σ_i denotes the ith strand crossing over the i+1st strand and σ_i^{-1} denotes the ith strand crossing under the i+1st strand. The string of sigmas then describes the given braid. With this notation in mind, we are ready to state the following proposition:

Proposition 1. *The closure of a n-string braid* $(\sigma_1\sigma_2\cdots\sigma_{n-1})^m$ *is a knot if and only if n and m are relatively prime. In this case, the closure of this braid is the T(m,n) torus knot.*

Proof. When we multiply braids together, we stack the sequence of crossings on top of itself. Note that $(\sigma_1\sigma_2\cdots\sigma_{n-1})$ takes each strand and moves it one position to the left with the first strand moving to the furthest right position. Thus, if we think of labeling each position as $1, 2, 3, ..., n$, then after one word, the strand that begins in position i will end in position $i-1(modn)$. Thus, after m-stackings, each strand will end m positions to the left or i-m(mod n).

If m and n are not relatively prime, then there exists $k \in \mathbb{Z}$ such that $k|m$ and $k|n$, i.e. $m = bk$ and $n = ak$ for some $a, b \in \mathbb{Z}$, $b < m$ and $a < n$. If we consider the strand that begins in position 0 and trace it through the closed braid, the first pass through ends at -m(mod n). Beginning here at the top of the braid, on the next pass through, what was the strand in position 0 ends at -2m(mod n), and then after another pass, (-3m)modn and eventually, strand 0 will be at position (-am)modn. Since m=bk, then -am(modn)= -abk(modn)= -bn(modn)= 0(modn). Thus, strand 0 ends up back in position 0. This creates a component of a link in the closed braid, proving that if m and n are not relatively prime, then we get a link and not a knot. Thus, we do not pass through all strands in this loop. If m and n were relatively prime, then we would pass through all of the strands.

Recall the definition of a $T(m, n)$ torus knot is a knot that wraps n times around a circle inside the torus and m times around a line through the hole in the torus.

If we place the center of our torus in the center of our braid closure, then the crossings in our braid will be m and the number of strands will be n.

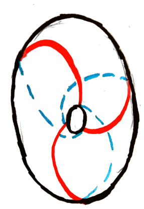
Figure 8:

□

2.3 The Bracket and Jones Polynomial

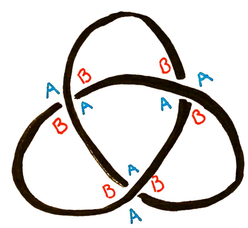
Let K be any knot. We can use the bracket polynomial, $< K >$, to calculate the Jones polynomial. The bracket polynomial may be calculated in the following

Figure 9:



manner. Looking at a diagram of a knot, we must first label the four spaces surrounding each crossing with A's and B's. To do so, we follow the following rules: the spaces counter clockwise from the over-strand will be labeled A and the spaces clockwise from the over-strand will be labeled B. See the example below with a labeling of the trefoil.

Figure 10:

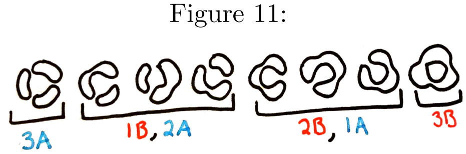


Second, we must calculate the number of states, where a state, s, is obtained by opening some sequence of A and B channels at each crossing. When we open an A channel, we open a passage way in the crossing between the two spaces marked with an A. When we open a B channel, we open a passage way in the crossing between the two spaces marked with a B. Since at each crossing there are 2 possible channels, there are $2^{c(k)}$ total states where c(k) denotes the number of crossings.

Third, we must find all possible states by splitting each crossing as an A split or a B split in all possible combinations. (See Figure 11)

Fourth, we must find $|s|$ by counting the number of circles in each split diagram.

Figure 11:



Finally, to find the bracket polynomial, we plug this information into the following formula.

$$< K >= \sum_{s \in \zeta} A^{a(s)} A^{-b(s)} (-A - A^{-2})^{|s|-1} \tag{1}$$

In the above formula, a(s) is the number of A channels open in each state, b(s) is the number of B channels open in each state s and ζ is the set of all possible states. This itself in not an invariant because it is changed by Reidemeister 1 move (see [Adams, 2004]). However if we adjust the formula slightly as follows, we do obtain a knot invariant known as the Laurent polynomial,f_k.

$$f_k = (-A^{-3})^{w(k)} < L > \tag{2}$$

Here $w(k)$ is the writhe and is calculated by subtracting the total number of negative crossings from the total number of positive crossings. Next to find the Jones polynomial, V_k, we must substitute $t^{-1/4}$ for A into the Laurent polynomial f_k.

This is one of several ways to calculate the Jones Polynomial. In the next section we will explore how the Trip matrix may be used to find $|s|-1$ and $w(k)$ in formula (1) and (2). This will allow us to compute the Jones polynomial using basic Linear Algebra.

2.4 The Trip Matrix and How to Compute It

The trip matrix was introduced by Louis Zulli [Zulli, 1995] in an attempt to calculate the bracket, Laurent and hence the Jones polynomial via Linear Algebra. In order to find the trip matrix of a diagram of a knot we must first adorn the diagram. At each crossing we place an arrow on each over-strand (in any fashion) and place an arrow on the under-strand so that if we rotate the over-strand arrow counterclockwise it will match the under-strand arrow. We also number the crossings, the ordering of which does not matter because the resulting Jones polynomial will be the same.

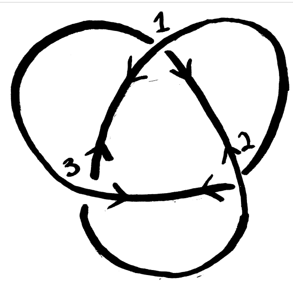
We will use this adornment to create a symmetric $n \times n$ matrix, T, filled with 0's and 1's where n is the number of crossings in the diagram. This matrix will be the trip matrix for this knot projection.

To fill in the T_{ij} entry, where i is the row and j is the column, we must use the following set of rules for each crossing.

If $i \neq j$, T_{ij} is defined as the number of times (mod 2) that a traveler passes through crossing i when starting at the overstrand of crossing j and traveling in the direction of the overstrand arrow. We stop tracing when we hit the crossing j again

For $i = j$, T_{ij} is defined as, if upon completing the trip above, the traveler finds the arrow on the under-strand urging him onward then $T_{ij} = 0$. If the traveler finds the arrow on the under-strand commanding him to go back then

Figure 12:



$T_{ij} = 1$. Note: The trip matrix T corresponds to the state where all A channels have been opened.

Given a state s, create a new matrix T_s from T, where, if a B-channel is opened at crossing i, the entry T_{ii} should be toggled from a 0 to 1 or vice versa. To find $< L >$ and f_k we use the main theorem from Zulli:

Theorem 2. *[Zulli, 1995] (a) The writhe of K is the number of zeros on the diagonal of T less the number of ones on the diagonal of T. (b) Suppose the state s is obtained from the state $AA \cdots A$ by toggling the labels in positions $i_1, i_2, i_3 \cdots i_m$. Let T_s be the matrix obtained from the matrix T by toggling the entries in the corresponding positions along the diagonal of T. Then:* $dim(NulT_s) = |s| - 1$

Using this theorem, we can easily calculate $< L >$, f_k and V_k as described in Section 2.4.

Example 1. *Consider the T(3,2) torus knot also known as the trefoil. Below is a chart with the trip matrix for the trefoil knot in figure 12 and the toggled matrices.*

State S	T_s	*dim Nul* T_s
AAA	$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$	*2*
BAA	$\begin{bmatrix}0&1&1\\1&1&1\\1&1&1\end{bmatrix}$	*1*
ABA	$\begin{bmatrix}1&1&1\\1&0&1\\1&1&1\end{bmatrix}$	*1*
AAB	$\begin{bmatrix}1&1&1\\1&1&1\\1&1&0\end{bmatrix}$	*1*
BBA	$\begin{bmatrix}0&1&1\\1&0&1\\1&1&1\end{bmatrix}$	*0*
BAB	$\begin{bmatrix}0&1&1\\1&1&1\\1&1&0\end{bmatrix}$	*0*
ABB	$\begin{bmatrix}1&1&1\\1&0&1\\1&1&0\end{bmatrix}$	*0*
BBB	$\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$	*1*

Remark 1. *A quick look at the Null Space dimensions and the number of circles created in the split diagram in Figure 11 shows they do in fact coincide. For the reader interested in seeing the general proof, see [Zulli, 1995]*

3 Results

With this background, we are now ready to examine the Jones Polynomial for T(2,n) torus knots. First, we show what the trip matrices look like for this family of knots.

Proposition 2. *Any T(n,2) torus knot will have an* $n \times n$ *trip matrix T with all 1's off the diagonal and either all 1's or all 0's on the diagonal.*

For example, for the T(3,2) knot, the corresponding trip matrix is either:

$$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix} \text{ or } \begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}. \tag{3}$$

Proof. By Proposition 1, we know that the closure of the 2-string braid $(\sigma_1)^n$ is the T(n,2) torus knot. The mirror image then will look like $(\sigma^{-1})^n$.

First, we must prove that the non-diagonal entries of the trip matrix will always be 1. We prove this by induction on the number of crossings. **Base Case**: Let $n = 3$. Note for $(\sigma_1)^n$ to be a non-trivial knot when closed it must be prime and greater than or equal to 3 (by Proposition 1).

Figure 13:

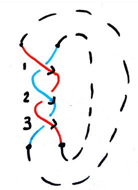
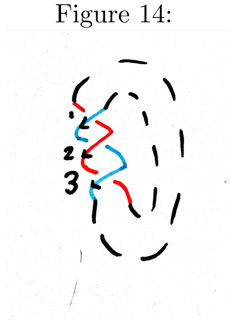
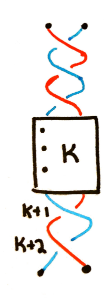


Figure 14:



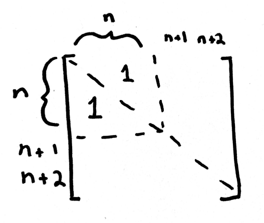
If we examine Figures 13 and 14, it is clear that starting at crossing j, each i crossing will be passed through once before returning back to j, giving a 1 in each T_{ij} entry for i and j. **Induction Case**: Assume that this is true for odd n, we must prove its true for $n + 2$. We add 2, so the number of crossings will remain odd.

Figure 15:



We know that the claim is true for the first n crossings. Thus, the matrix looks like

Figure 16:



In Figure 15, it is shown that $k+1$ and $k+2$ will also be passed through only once. Therefore, the crossing in k will also only be passed through once.

Next, we show that the diagonal entries of the trip matrix will always be either all 1's or all 0's. If we draw a T(2,n) braid and pick the crossing orientation so that the over-crossings are oriented downward.

Figure 17:

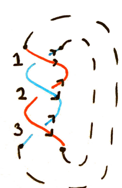
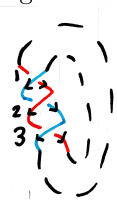


Figure 18:



This will force all of the under-crossings to be oriented upward or downward if it is the mirror image knot, following the counterclockwise rule. This makes it so that the diagonal will either be all 1's or all 0's because when we trace the knot to create our trip matrix when we get back to our original crossing all the arrow will be uniformly urging us backward or onward.

□

Corollary 1. *The writhe of any T(n,2) matrix is either n or -n.*

Proof. By Theorem 2 (Zulli's Theorem) w(T(n,2)) = the number of 0's on the diagonal less the number of 1's on the diagonal. The previous proposition guarantees either all 1's or all 0's. □

Lemma 1. *Let T be a trip matrix for T(2,n) where all diagonal entries are 1's. The dimension of the null space for each toggled matrix, T_s will be $n-(b+1)$ where $b<n$ is equal to the number of B channels open in that particular state s. When all of the B channels are open, the dimension of the null space of T_s will be 1.*

Proof. By Proposition 2, the matrix T consists of all 1's **Case 1** Suppose $b<n$. We know that from the Rank-Nullity Theorem (see for example [Lay and McDonald, 2016]), that $n = \dim \text{Col } T_s + \dim \text{Nul } T_s$. All of the columns that are not toggled will be identical because they will be all 1's. Thus, the dimension of the column space will be at most b + 1 where b is the number of columns that have been toggled and are thus not identically 1. We claim, in fact that dim Col $T_s = b+1$, since the nonidentical columns of the matrix are linearly independent.

Suppose $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_b$ are the toggled column vector in T_s. Each $\vec{v}i$ will consist of all 1's except one 0 entry. Because the matrix T_s is obtained by toggling only along the diagonal, then each $\vec{v}_i$ will have a 0 in a different row. Furthermore, since $b<n$, there is at least one row that has a 1 entry for each $\vec{v}_i$. Now, let

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_b\vec{v}_b \equiv \vec{0}(mod2). \tag{4}$$

Suppose, for contradiction that $a_i \neq 0$ for some i. If b is odd then some even collection of the a_i must be 1 and the remaining a_i must be 0 to ensure that the row with all 1's in each $\vec{v}_i$ will be even (hence 0 mod 2). Suppose, WLOG that $a_1, a_2, \cdot, a_n = 1$ where n is even and $a_{n+1} \cdots a_b = 0$. Choose the row in $\vec{v}_1$ that has the 0. Then, $\vec{v}_2 \cdots \vec{v}_b$ all have 1's in that row. Thus, $a_2\vec{v}_2 + \cdots + a_n\vec{v}_n \equiv 1$ since the sum is odd and $a_{n+1}\vec{v}_{n+1} + \cdots + a_b\vec{v}_b \equiv 0$, thus, $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_b\vec{v}_b \neq 0$. Thus, we cannot have a non zero coefficient. A similar argument works if b is even, thus proving $a_1, ...a_b = 0$ and $\vec{v}_1 \cdots \vec{v}_b$ are linearly independent.

Case 2: Suppose b = n. Thus, there are only zeros on the diagonal. In this case, the last column will be a linear combination of the other two.

$$\begin{bmatrix}0\\1\\1\\1\\\vdots\\1\end{bmatrix} + \begin{bmatrix}1\\0\\1\\1\\\vdots\\1\end{bmatrix} + \begin{bmatrix}1\\1\\0\\1\\\vdots\\1\end{bmatrix} + \cdots + \begin{bmatrix}1\\1\\1\\\vdots\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\\\vdots\\0\end{bmatrix}. \tag{5}$$

Now, the bottom number of 1's will be even, so when added together mod 2, we will get a 0. Every other row will have an odd number of 1's resulting in a 1 when the sum is taken. Thus, the last column is in fact a linear combination of the first n-1 columns. Thus, dim Col $T_s \leq n-1$ and is in fact equal to n-1 since the remaining vectors are the type in case 1. Thus, dim Nul $T_s = n-(n-1) = 1$. $\square$

With the form of the trip matrix and the associated dimensions of the null space for the toggled matrices, we have all the necessary ingredients to compute V_k for $T(2,n)$ torus knots. The proof consists solely of Linear Algebra and basic algebra manipulations. We do, however, need a basic combinatorial result, the Binomial Theorem.

Theorem 3 (Binomial Theorem)**.**

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \tag{6}$$

Proposition 3. *For the T(2,n) torus knots that have all 1's along the diagonal in the trip matrix where* $3 \leq n \leq 1000$*,*

$$V_k = t^{1-n/2}(1 + t^{-2} - t^{-3} + t^{-4} - \cdots + t^{-n}). \tag{7}$$

Proof. Let K = T(2,n) torus knot with a trip matrix with all 1's, where n is an odd number.

Recall that

$$< K >= \sum_s A^{a(s)} B^{b(s)} d^{|s|-1}.$$

Suppose s is a state with k' B-channels open. Note that there are $\binom{n}{k'}$ such states since this is the number of possible ways to choose k' crossings out of n total crossings in which to open a B-channel. By Theorem 2, for this state s, $|s|-1 = $ dim Nul (T_s) where T_s is the toggled matrix. By Lemma 1, dim Nul$(T_s) = n-(k'+1)$, for k' in $\{0, 1, ...n-1\}$ and dim Nul$(T_s) = 1$ if $k' = n$. Thus, the formula for $< K >$ becomes,

$$< K >= \binom{n}{0} A^n d^{n-1} + \binom{n}{1} A^{n-1} B^1 d^{n-2} + \binom{n}{2} A^{n-2} B^2 d^{n-3} + \cdots + \binom{n}{n-1} A^1 B^{n-1} d^0 + \binom{n}{n} B^n d^1.$$

First, we multiply by $(-A^{-3})^{w(k)}$ which gives f_k. Recall $w(k) = -n$ by Corollary 1. This gives us,

$$f_k = -A^{4n} d^{n-1} - A^{3n+n-1} B^1 d^{n-2}(n) - A^{3n+n-2} B^2 d^{n-3} \binom{n}{2} + \cdots - A^{3n} B^n d^1.$$

Then substituting in $B = A^{-1}$ gives,

$$f_k = -A^{4n} d^{n-1} - A^{4n-2} d^{n-2}(n) - A^{4n-4} d^{n-3} \binom{n}{2} + \cdots - A^{2n} d.$$

Finally, we substitute $t^{-1/4}$ for A which gives us,

$$f_k = -t^{-n}d^{n-1} + (-t^{-n+1/2})d^{n-2}(n) + (-t^{-n+1})d^{n-3}\binom{n}{2} + \cdots + (-t^{-1/2})d.$$

Then, we substitute $d = -A^2 - A^{-2}$ and $t^{-1/4}$ which gives,

$$\begin{aligned}
&V_k = -t^{-n}(-t^{-2/4} - t^{2/4})^{n-1} \\
&+ (-t^{-n+1/2})(-t^{-2/4} - t^{2/4})^{n-2}(n) \\
&+ (-t^{-n+1})(-t^{-2/4} - t^{2/4})^{n-3}\binom{n}{2} + \cdots + (-t^{-1/2})(-t^{-2/4} - t^{2/4}).
\end{aligned}$$

Using the Binomial Theorem, we simplify to,

$$V_k = (-1)^{n-1}\sum_{k}^{n-1}\binom{n-1}{k} - t^{-3n/2+1/2+k} \tag{8}$$

$$+ (-1)^{n-2}(n)\sum_{k}^{n-2}\binom{n-2}{k}2 - t^{-3n/2+3/2+k} \tag{9}$$

$$+ (-1)^{n-3}\binom{n}{2}\sum_{k}^{n-3}\binom{n-3}{k} - t^{-3n/2+5/2+k} \tag{10}$$

$$+ (-1)^{n-4}\binom{n}{3}\sum_{k}^{n-4}\binom{n-4}{k} - t^{-3n/2+7/2+k} + \cdots \tag{11}$$

$$+ (-1)^{n-(n-1)}\sum_{k}^{n-(n-1)}\binom{n-(n-1)}{k} - t^{-3n/2+n-1/2+n/2+k} \tag{12}$$

$$- \binom{n}{n-1}t^{-1/2n-1/2} + t^{-1-n/2} + t^{1-n/2}. \tag{13}$$

All that remains to be shown, is that this simplifies to the stated Jones Polynomial. First, notice that the lowest degree term is $t^{-3n/2+1/2}$ which is obtained from the first sum. The coefficient by setting k=0, for which we get +1 on $t^{(-3n/2+3/2)}$, is obtained from the setting k=1 in the first sum and k=0 in the second sum. Continuing in this fashion, we see that the coefficient on $t^{(-3n/2+(2M+1)/2)}$ is obtained from the first M+1 sums. This coefficient is $\sum i = 0^M \binom{n-=0+1)(-1)^M}{M-i} * \binom{n}{i})$. Using a computer, we computed this for all $0 \le M \le 1000$ (See appendix for Code). This sum is 1 if M is odd and -1 if M is even. Thus,

$$V_k = t^{(3n/2+1/2)} + t^{(-3n/2+3/2)} - t^{(-3n/2+5/2)} + ... + t^{(-n/2+1/2)}. \tag{14}$$

Factoring out $t^{1/2-n/2}$ gives the result.

□

Corollary 2. *The Jones polynomial of the other T(2,n) torus knot is* $V_k = t^{n/2-1}(1 + t^2 - t^3 + t^4 \cdots - t^n)$

Proof. Replace t by t^{-1} in the previous proposition. □

4 Further Directions

In Adams *The Knot Book* [Adams, 2004] he poses an open question about trying to find a more elementary proof of the explicit formula for T(m,n) torus knots. In this paper we tackled a small portion of this question. We would like to come up with a more general proof of the coefficients being equal to -1 or 1 in the proof of Proposition 3 so that we can get the formula for the Jones polynomial of T(2,n) torus knots for all n. This research can be taken further to generalize T(2n,m) torus knots, with the goal of building up to the proof of T(m,n) torus knots.

5 Appendix

Below is the Python code used in the proof of Proposition 3:

```
def binom(n, r):
  p = 1
    for i in range(1, min(r, n - r) + 1):
      p *= n
      p //= i
      n -= 1
    return p

  for k in range(3, 1000, 2):
    print("k= " + str(k))
    for x in range(1, k):
      rowList = []
      for i in range(x, 0, -1):
        currentCo = (-1)**i * binom(k, (i-1)) * binom((k-i), (x-i))
        rowList.append(currentCo)
      printSum = sum(rowList)
      print(printSum)
```

6 Bibliography

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