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Polynomial Knot Invariants

An Exploration of the Alexander Polynomial for Knots

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Abstract

What is a knot? Imagine a string, which we tie arbitrarily, and then fuse its two free ends together to form a closed loop. In technical language, a knot K is an embedding $f : S^1 \to \mathbb{R}^3$. This paper investigates the first polynomial knot invariant, the Alexander polynomial, introduced by the American mathematician James Waddell Alexander II in 1923. We examine the Alexander polynomial of torus knots via two computing methods and the concrete form of this polynomial for torus knots. We then compare these polynomials and show the uniqueness of the Alexander polynomial for each torus knot, up to mirror images. Finally, we conclude with a Theorem from Stoimenow [1] that the Alexander polynomial of a closed $3-$braid is never 1, and prove it for four or less terms in the braid word.

Contents

1 Introduction

Traditionally, it is said that there are as many sailing knots as there are stars in the night sky, which can make the job of a Knot Theorist to differentiate them quite elaborate. Although tying knots dates back to prehistoric times, their mathematical associations only emerged at the end of the 18^{th} century with the French mathematician Alexandre Vandermonde. However, mathematical studies of knots only began in the 19^{th} century with the German mathematician Carl Friederich Gauss, and it was not until 1923 that the American mathematician James Waddell Alexander II devised an initial theory of differentiating certain knots from one another.

Mathematics plays an important role in distinguishing two knots as we can associate to knots mathematical objects (such as numbers, polynomials, groups, etc.) that, when different, guarantee the knots are distinct. These objects are known as invariants. The goal of this project is to study the first polynomial knot invariant, the Alexander polynomial. In particular, we explore a more elementary method to compute the Alexander polynomial of torus knots and prove the formula formally in two cases:

1. $T(n, 2)$;
2. $T(n, n-1)$.

We use this formula to prove that the trivial torus knot is the only torus knot with trivial Alexander polynomial, and to classify all torus knots. We also classify the general knots with respect to their braid index and develop a procedure to determine the form of the Alexander polynomial for knots with braid index three. We evaluate the first few cases with respect to the number of generators in the braid word of each knot, and conjecture the form of the Alexander polynomial for knots with braid index three. Last, but not least, we investigate the difficulties that arise when applying this procedure to more general cases, and to knots with braid index larger than three. We quickly recognized the need for computer assistance in developing the computations.

We begin this paper with an overview of the basic concepts of knot theory needed for this project, including knots, links, braids and knot invariants. We then explore the Burau representation of B_n, the braid group on n strings, in order to use the power of linear algebra to study B_n. We then introduce torus knots $T(p, q)$, which are knots embedded on the surface of a torus, and investigate the form of the braid word for torus knots. Afterwards, we present the Alexander polynomial and two ways of computing it, via the defining matrix or via the Burau matrices. We also included a section for Future Directions and Open Questions, in which we discuss the complexity of these topics, and some of the many questions that are still unanswered by the mathematical community.

2 Background

The purpose of this section is to introduce basic concepts of knot theory needed for this project. We begin by describing knots and links, followed by defining the concept of a knot invariant as a tool for differentiating knots. We then explore two main objects in this research: torus knots and braids. For more information on these topics or knot theory more broadly, see [2] and [3].

2.1 Knots and Knot Invariants

What is a knot? Imagine a string, which we tie around itself arbitrarily, and then fuse its two free ends together to form a closed loop. This is what we will call a knot. A more technical definition is presented below.

Definition 2.1. A **knot** K is an embedding $f : S^1 \to \mathbb{R}^3$.

A noteworthy aspect of knots is that the string never intersects itself; rather, the string either **under-crosses** or **over-crosses** itself. Since knots live in three dimensions, we can represent them in two dimensions through projections.

Definition 2.2. A **projection** of a knot K is a $2-$dimensional representation of the knot, where under-crossings are represented by breaks in the strands.

The simplest mathematical knot is called the **unknot**, or the **trivial knot**, and it can be obtained by taking the free ends of a string and fusing them together with no crossings. Thus, a projection of the unknot would be a circle. However, this projection is not unique, and classifying all projections of the unknot still remains an open question. One projection of the unknot distinct from the circle was discovered by the German mathematician Lebrecht Goeritz in 1934. This projection is portrayed in Figure 2.1.

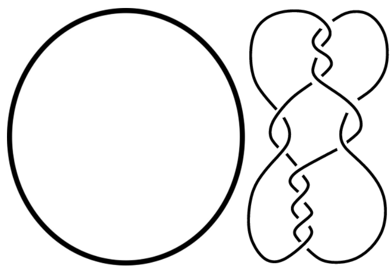

Figure 2.1: Two Projections of the Unknot.

Definition 2.3. A **link** is a collection of finitely many knots that do not intersect each other, but which can be knotted together. Each link has components, represented by the knots that make up the link.

The focus of this paper is knots, but links will come up when we discuss specifics of torus knots and in the last chapter. Figure 2.2, which resembles the logo of the Olympic Games, is an example of a link with three trivial components, known as the Borromean Rings.

Figure 2.2: The Borromean Rings.

In general, it is possible for two knots to look different but be deformed to look identical.

Definition 2.4. Two knots K_1 and K_2 are said to be **equivalent knots** if one can be continuously deformed to look like the other. This deformation is formally called **ambient isotopy**.

Looking back at Goeritz' projection of the unknot in Figure 2.1, it is not very clear how to deform it into a circle. This is the case generally for two projections of the same knot, which led mathematicians to seek some sort of mathematical objects to differentiate knots and projections of the same knot.

Definition 2.5. A **knot invariant** is an object that remains unchanged under continuous deformations.

Thus, any two equivalent knots must have the same invariants. Sadly, for the invariants known up to this point, if two projections have the same invariant, we cannot conclude that they are equivalent knots. An example of a knot invariant is shown below.

Example 2.1. *For any knot K, the* ***unknotting number*** $u(K) = n$ *if there exists a projection of K in which changing n crossings would turn K into the unknot, and there is no projection of K for which this is true by changing fewer crossings.*

For example, the unknotting number of the trefoil knot (see Figure 2.5) is 1; that is, changing any of the three crossings in the trefoil would result into the unknot. The American mathematician Martin Scharlemann proved in 1985 that if a knot has unknotting number 1, then it must be a prime knot (i.e. it cannot be expressed as the composition of two non-trivial knots) [2]. This is a great example of how we can use the unknotting number to distinguish knots, but it also hints that we cannot use this invariant to highlight them. Many other invariants exist, and we will focus on the Alexander Polynomial (see Chapter 3).

2.2 Braids

2.2.1 Overview

Braids turn out to be effective ways to remodel knot projections into (usually) more organized pictures. Imagine n vertical strings, or strands, attached on two parallel solid rods. A braid on n strands is essentially an intertwining of the n strands, with the properties that no two strands intersect, and any parallel line to the two rods that is contained between them intersects each strand exactly once. This last property basically reads that no string "turns back up".

We would like an efficient way to describe braids, which can be done in the following way. To the strands we may assign one of the operations σ_i or σ_i^{-1}, defined in Figure 2.3, where $i = \overline{1, n-1}$.

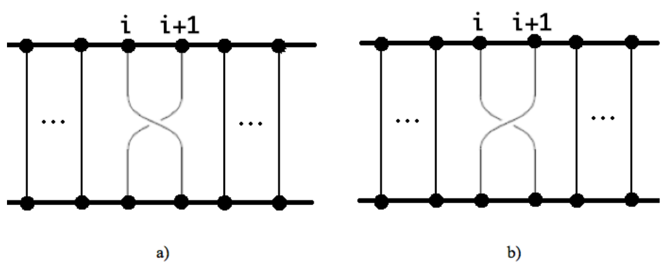

Figure 2.3: Definition of σ_i (left) and σ_i^{-1} (right).

The operation σ_i is defined by crossing the i^{th} strand over the $(i+1)^{th}$ strand; similarly, the operation σ_i^{-1} is defined by crossing the i^{th} strand under the $(i+1)^{th}$ strand. Each $n-$braid can be expressed with a **braid word** using the generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$, or their inverses, $\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_{n-1}^{-1}$. For example, the Borromean Rings in Figure 2.2 has the braid word $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$.

The relevance of braids is that if we glue together the top and bottom rods, we always form a knot or a link, called the **closure** of the braid. This implies that every closed braid is a knot or a link. Somewhat surprisingly, the reverse is also true.

Theorem 2.1 (Alexander, 1923)**.** *Every knot and link is a closed braid.*

The set of all braids on n strands is in fact a group, and we will devote Section 2.2.2 for further study of this group.

Definition 2.6. The **braid group** on n strings, denoted B_n, is defined as the set of all braids with n strands, along with composition of braids, $\bigoplus$. As noted in [4], this operation is defined "by joining the bottom points of the first braid to the top points of the second", as seen in Figure 2.4.

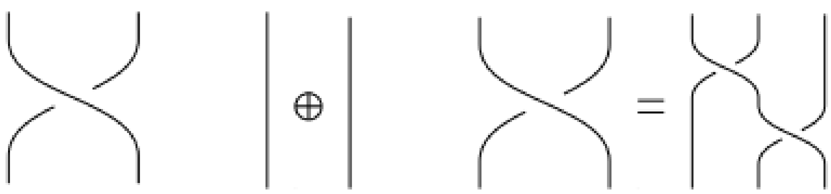

Figure 2.4: Example of Braid Composition.

Alternatively, the group B_n may also be defined using the aforementioned generators $\sigma_1, \ldots, \sigma_{n-1}$, and some relations involving them.

Definition 2.7 (Algebraic)**.** The group B_n is generated by $\{\sigma_1, \ldots, \sigma_{n-1}\}$ with the following relations:

1. $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geq 2$.

2. $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \leq i \leq n-2$.

2.2.2 Burau Representation

In this section, we seek a group representation of B_n. The reason for this is because we would like to describe the elements of the braid group as invertible matrices, in order to use the power of linear algebra to study B_n.

To do so, let $D = \{z \in \mathbb{C} : |z| \leq 1\}$ be the unit disk in the complex plane, and let $D_n = D - \{z_1, z_2, \ldots, z_n\}$, where $\{z_i | i = \overline{1, n}\}$ is a collection of points in D. Let α be any loop in D_n based at $z_0 \in D_n$ and let $\phi\alpha$ be the **total winding number** for α with respect to $z_1, z_2, \ldots, z_n$, which is defined as the unique number of times α winds around the n points. For any loop α, $\phi\alpha$ is a positive integer and we obtain a homomorphism $\phi : \pi_1(D_n, z_0) \to \mathbb{Z}$, where $\pi_1(D_n, z_0)$ is the **fundamental group** of D_n, whose elements are the equivalence classes under homotopy of the loops in D_n. Recall the following definitions from Algebraic Topology. For more details, see [5].

Definition 2.8. Let $p : E \to B$ be a map. If f is a continuous mapping of some space X into B, a **lifting** of f is a map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$.

Definition 2.9. Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map. Define $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by the equation $h_*([f]) = [h \circ f]$. The map h_* is called the **homomorphism induced by** h, relative to the base point x_0.

The following lemma from [3] justifies the reason why the corresponding matrices have $'t'$s in their entries.

Lemma 2.2. *Let $H_1(\tilde{D_n})$ be the first homology group of a regular covering space of D_n. Then $H_1(\tilde{D_n})$ is a free $\mathbb{Z}[t, t^{-1}]$ module.*

Consider now the homomorphism $\psi_r : B_n \to GL(H_1(\tilde{D_n}))$ given by $\psi_r : \beta \mapsto \tilde{h}_*$, where GL denotes the **general linear group**, whose elements are $n \times n$ invertible matrices, and $\beta \in B_n$. This homomorphism is well-defined [3], and we shall call it the reduced **Burau Representation** of B_n, named after the German mathematician Werner Burau. The paper *Braid Group Representations* [3] gives a thorough derivation of the matrices for the reduced representation via level diagrams. However, the final results are sufficient for the purpose of this paper. If $n \geq 3$, then

$$\psi_r\sigma_1 = \begin{bmatrix} -t & 1 & \\ 0 & 1 & \\ & & I_{n-3} \end{bmatrix}, \psi_r\sigma_{n-1} = \begin{bmatrix} I_{n-3} & & \\ & 1 & 0 \\ & t & -t \end{bmatrix}$$

and for $2 \leq i \leq n-2$,

$$\psi_r\sigma_i = \begin{bmatrix} I_{i-2} & & & & \\ & 1 & 0 & 0 & \\ & t & -t & 1 & \\ & 0 & 0 & 1 & \\ & & & & I_{n-i-2} \end{bmatrix}.$$

The operation $\bigoplus$ describing braid composition becomes matrix multiplication. The blank entries in the matrices are zero entries. The Appendix of this paper proves that the braid relations in Definition 2.7 are satisfied for these matrix representations.

2.3 Torus Knots

A **torus** is defined mathematically as the Cartesian product $S^1 \times S^1$, or as we all know it, the surface of a doughnut. This section aims to examine torus knots, which are knots that are embedded on the surface of a torus. Figure 2.5 portrays the so-called trefoil knot, in space and on a torus.

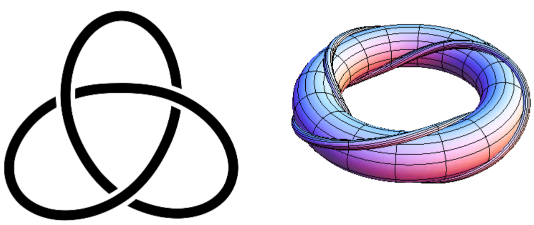

Figure 2.5: The trefoil knot in space (left) and on a torus (right).

The **inner** and **outer equators** of the torus are the small and large circles around the central void of the torus, respectively. Define the **toroidal direction** along the torus as the direction about the equators, and the **poloidal direction** as the direction along a small circle around the surface of the torus. A **longitude curve** is a curve that wraps once along the toroidal direction, and a **meridian curve** is defined as a curve that wraps once along the poloidal direction. Due to the lack of over- and under-crossings, a knot will travel along the torus meridionally and longitudinally an integer number of times.

Definition 2.10. A $\boldsymbol{T(p,q)}$ **torus knot** intersects a meridian curve q times and a longitudinal curve p times.

Note that by this definition, it also follows that a torus knot $T(p,q)$ wraps p times meridionally and q times longitudinally around a torus. The same definition holds for torus links as well, and we shall distinguish torus knots from torus links in Theorem 2.4.

Example 2.2. *The trefoil knot on a torus in Figure 2.5 wraps around the torus three times meridionally, because it crosses the longitude three times, and twice longitudinally, because it crosses the meridian twice. For this reason, we call it a* $(3,2)-$*torus knot, or simply* $T(3,2)$.

Note that if either p or q equals 1, we obtain the trivial torus knot (i.e. the unknot). Our ultimate goal in Chapter 4 is to compute the Alexander polynomial of torus knots. We illustrate two theorems and their proofs that shed more light on the behavior of torus knots, both essential in later computations and analysis. Note that in this paper, we do not distinguish left-handed and right-handed knots.

Theorem 2.3. *Suppose that p and q are positive integers such that $T(p,q)$ and $T(q,p)$ are knots. Then they must be equivalent knots.*

Proof. Consider the torus knot $T(p,q)$, and remove a small disk from the torus that does not intersect the knot at any point, whilst keeping the boundary of the disk on the torus. The circle that represents the boundary of the disk thus becomes a boundary for the torus too, and the new shape T_1 can be deformed into two bands that are attached to each other. The long band, which corresponds to a longitude of the torus, wraps around the smaller band, which corresponds to a meridian of the torus. Since T_1 contains the knot, the deformation will carry it along. Take the long band and turn it inside out; the new figure has the two bands attached, but the short band is now outside the long band. Now take the short band and turn it inside out as well; again, we have the long band wrapping around the small band, but each of them rotated 90° in the process. We can deform the bands to another torus with a circle boundary, T_1', with the distinction that the band in

T_1 that corresponds to the longitude of the torus is the band in T_1' that corresponds to the meridian, and the band in T_1 that corresponds to the meridian of the torus in the band in T_1' that corresponds to the longitude. Therefore, filling in the boundary of T_1' gives us the torus $T(q,p)$, which completes our proof. Figure 2.6 found in [2, p. 150] demonstrates this process visually for a trefoil torus knot.

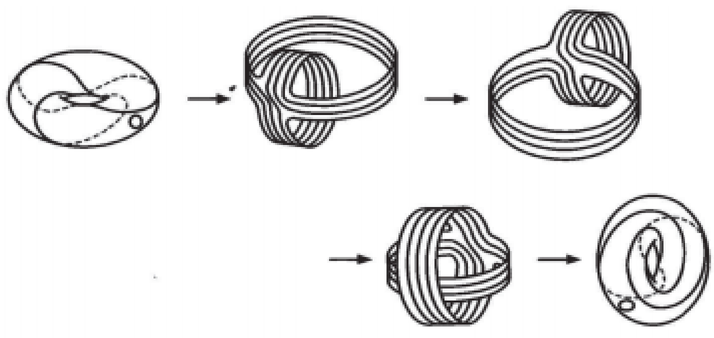

Figure 2.6: Illustration that a $(3,2)-$torus knot is a $(2,3)-$torus knot.

□

Corollary 2.3.1. *Suppose that p and q are positive integers such that $T(p,q)$ and $T(q,p)$ are links. Then they must be equivalent links.*

Proof. The proof of this corollary follows analogously from the proof of Theorem 2.3. □

An essential consequence of the previous theorem and corollary is that we can always assume that either $q \leq p$ or $p \leq q$.

Theorem 2.4. *If p and q are co-prime positive integers, then $T(p,q)$ is a knot; otherwise, it is a link.*

Proof. Suppose that $p \geq q$. The proof relies heavily on the construction of a torus knot $T(p,q)$. Consider p points $A_0, A_1, \ldots, A_{p-1}$ written clockwise along the inner equator and points $B_0, B_1, \ldots, B_{p-1}$ written clockwise along the outer equator. For each $i = \overline{0, p-1}$, we draw a strand between A_i and B_i across the bottom of the torus, and a strand between A_i and B_{i+q} across the top of the torus, where the numbering of the indices is modulo p. We claim that this construction describes the knot, or link, $T(p,q)$.

First, note that the clear distinction between the bottom and top strands ensures no under- or over-crossings. Consider the case when p and q are co-prime. Begin with A_0, and move along the top strand to B_q, then along the bottom strand to A_q. Now move along the top strand to B_{2q} and along the bottom strand to A_{2q}. We repeat this process until arriving to $A_{(p-1)q}$, from where we move along the top strand to B_{pq} and along the bottom strand to A_{pq}, which are, in fact, B_0 and A_0. Since p and q are co-prime, the collection $\{1 + iq : i = \overline{0, p-1}\}$ of elements living in $\mathbb{Z}_p$ does not have repeated elements. This is because, supposing otherwise, then $1 + iq \equiv 1 + jq \pmod{p}$ for some $i, j = \overline{0, p-1}, i \neq j$, which implies the contradiction $(i - j)q|p$. Therefore, we began and ended the process described above in the same point A_0 and traveled through each point on the inner and outer equator once. This continuity foreshadows a knot. If each step of the process is described by moving from a point on an equator to a point on the other equator, then there are $2p$ steps. Also, we begin on the inner equator and every two steps, we arrive back on it. This means that throughout the process, we cross the meridian p times, which implies that the continuous strand wraps around the torus p times longitudinally. For simplifying the next argument, we dismiss the modular arithmetic in the numbering of the point indices. For crossing the longitude once, we need to pass p points and arrive at the initial point A_0. Since the final point in the process is A_{pq}, this happens precisely $\frac{1}{p}(pq) = q$ times, which implies that the continuous strand wraps around the torus q times meridionally. We thus obtain the knot $T(q, p)$, which by Theorem 2.3 is equivalent to the knot $T(p, q)$.

Now consider the case when p and q are not co-prime, and let $d = \gcd(p, q)$ and $p_0 = \frac{p}{d}$, which implies that $\gcd(p_0, q) = 1$. For each $k = \overline{0, d-1}$, we apply the same process as above to A_k, with the difference that we end at A_{k+p_0q}. Since p_0 and q are co-prime, then by a similar argument as in the previous case, the collection $\{k + iq : i = \overline{0, p_0 - 1}\}$ does not have repeated terms. Also note that

$$k + p_0q = k + \frac{p}{d}q = k + p\frac{q}{d} \equiv k \pmod{p},$$

which ensures that each process begins and ends in the same point. We wish to show that each point A_i on the inner equator appears in a unique process. Since the d processes each contain p_0 points, then we travel along a total number of $dp_0 = p$ points, and thus it is sufficient to prove that each point A_i appears in at least one process. Let i be a positive integer between 0 and $p-1$, and apply the Division Theorem to write $i = di_0 + k$,

$k < d$. Since $i < p$, then $i_0 < p_0$ and this implies that A_i is part of the process that begins at point A_k. We conclude that the d continuous strands obtained from each process is disjoint from one another. Analogously with the previous case, the continuous strands wrap around the torus a total number of p times longitudinally and q times meridionally, and we thus obtain the link $T(q,p)$, which by Corollary 2.3.1 is equivalent to the link $T(p,q)$. □

From this point on, we always consider that p and q are co-prime, unless otherwise stated. Note a consequence of the proof of the second case of the last theorem; if $p = q$, we obtain p components of the link, each a strand between A_i and B_i, where $i = \overline{0, p-1}$. This is the trivial link with p components.

2.3.1 Braid Word for Torus Knots

Not only did Werner Burau find a concrete expression for the braid generators, but he also obtained a formula for the Alexander polynomial of a knot, which is discussed in Chapter 3. This formula relies on his earlier findings and the braid word of a knot, which behaves elegantly for torus knots.

Theorem 2.5. *The braid word of a torus knot $T(p,q)$ is $(\sigma_1\sigma_2\cdots\sigma_{p-1})^q$.*

Proof. Consider the braid B with braid word $(\sigma_1\sigma_2\cdots\sigma_{p-1})^q$, where p and q are co-prime positive integers. We will prove that this braid word describes the torus knot $T(p,q)$. Note that when we close the knot, a vertical bar represents the longitude and the horizontal bars represent a latitude for the torus. We begin by denoting the p points on the upper rod by $A_1, A_2, \ldots, A_p$. After one iteration of the generators $\sigma_1, \sigma_2, \ldots, \sigma_{p-1}$, A_1 ends up in position A_p because each σ_i shifts A_1 from position A_i to position A_{i+1}. This represents crossing the longitude once. Since the iteration appears q times, then the resulting knot crosses the longitude q times. Moreover, when we close the strands in order to form the knot, each of the p strands will intersect a meridian curve. This means that the resulting knot intersects a meridian curve p times, and by the definition of a torus knot, we obtain $T(q,p)$. By Theorem 2.3, this is equivalent to the torus knot $T(p,q)$, and our proof is complete. Note that $T(p,q)$ is indeed a torus knot and not a link by Theorem 2.4. □

3 Alexander Polynomial

The Alexander polynomial was introduced by the American mathematician James Waddell Alexander II in 1923 as the first polynomial knot invariant. Essentially, the Alexander polynomial assigns a polynomial with integer coefficients to any knot. This can be done in several ways; through the defining matrix, a method introduced by Alexander himself in 1928, via the matrices given by the Burau representation, or the more modern approach via Knot Floer Homology, just to name a few. In this chapter, we aim to introduce the first two methods aforementioned. One property of the Alexander polynomial that is very important in our later analysis is that the polynomial remains unchanged up to multiplication of $\pm t^n$, for any integers n. One weakness of the Alexander polynomial is that it does not detect mirror images; the **mirror image** of a knot is the knot obtained by changing the over-crossings to under-crossings and the under-crossings to over-crossings. As we mentioned before, we shall not consider mirror images when comparing knots or their polynomials.

3.1 Computing via Defining Matrix

The aim of this section is to present one original definition of the Alexander polynomial, introduced by J. W. Alexander in his 1928 paper [6]. We begin by identifying the crossings, or nodes, of some knot K in the knot diagram, and label all regions adjacent to them. It is noteworthy to mention here that the number of regions will always differ from the number of nodes by two. Consider a clockwise orientation. At each node, we draw two dots on the left of the under-crossing, one above and one below the over-crossing, as portrayed in Figure 3.1. To each such crossing, we associate the equation

$$xA - xB + C - D = 0,$$

and the coefficients of this equation are referred to as **vertex weights**. Nevertheless, the following result in [7] allows us to disregard the alternation of the signs in the equation.

Lemma 3.1. *Consider the crossing and the labeling in Figure 3.1. Then switching the vertex weights to* $\{x, x, 1, 1\}$ *only changes the Alexander polynomial by a global sign.*

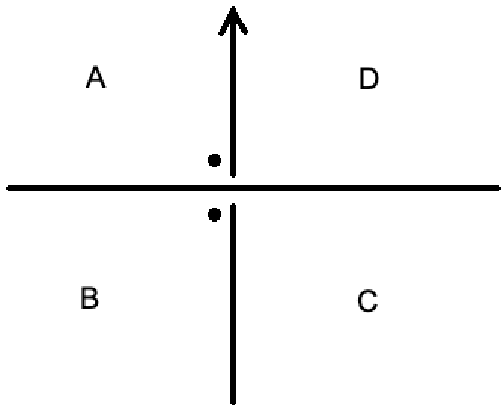

Figure 3.1: Labeling the Regions at each Node.

Since the Alexander polynomial remains unchanged up to multiplication of $\pm t^n$, this means that we can instead associate the equation

$$xA + xB + C + D = 0$$

to the crossing. Now let $R_1, R_2, \ldots, R_s$ be the regions in the knot diagram of K. After identifying each of the $s-2$ nodes, we consider the system of $s-2$ equations determined by the method presented above. Then we associate a matrix M_K to the knot K, in which each column i corresponds to the region R_i. That is, the j^{th} entry in the i^{th} column is given by the coefficient of R_i in the j^{th} equation in our system of equations. For any two regions R_j and R_k in the knot diagram, let $M_K[R_j, R_k]$ denote the matrix obtained from M_K by deleting the j^{th} and k^{th} columns. Then the Alexander polynomial can be defined as

$$\Delta_K(x) = \det(M_K[R_j, R_k]),$$

for any j and k such that R_j and R_k are adjacent regions.

Example 3.1. *We now illustrate how this method works on the trefoil knot. We first identify the nodes in its knot diagram and label its regions, as portrayed in Figure 3.2. Note that E is the region outside all of the knot.*

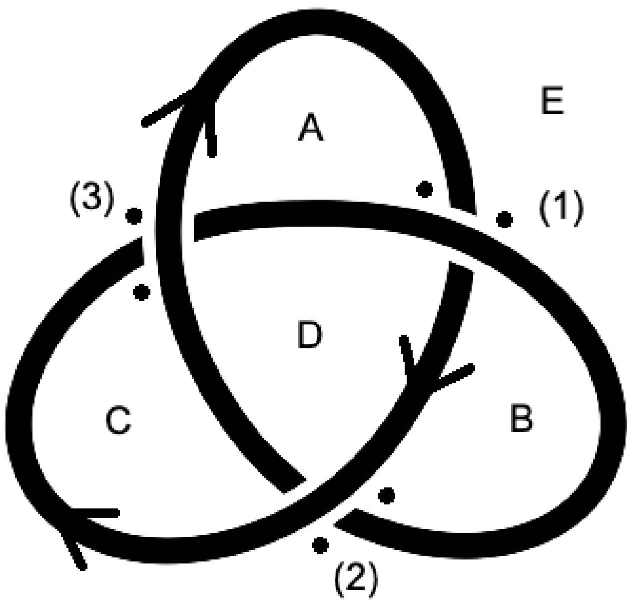

Figure 3.2: Labeling the Trefoil Knot.

From this, we obtain the following system of three equations:

$$\begin{cases} xA + B + D + xE = 0 \ (1) \\ xB + C + D + xE = 0 \ (2) \\ A + xC + D + xE = 0 \ (3) \end{cases}$$

This implies that

$$M_K = \begin{bmatrix} x & 1 & 0 & 1 & x \\ 0 & x & 1 & 1 & x \\ 1 & 0 & x & 1 & x \end{bmatrix}.$$

We pick the adjacent regions A and E, and write

$$M_K[A, E] = \begin{bmatrix} 1 & 0 & 1 \\ x & 1 & 1 \\ 0 & x & 1 \end{bmatrix}.$$

By applying co-factor expansion along the first row, we obtain

$$\Delta_K(x) = \begin{vmatrix} 1 & 1 \\ x & 1 \end{vmatrix} + \begin{vmatrix} x & 1 \\ 0 & x \end{vmatrix} = 1 - x + x^2,$$

and we conclude with the Alexander polynomial of the trefoil knot.

3.2 Computing via Burau Matrices

The aim of this section is to introduce another method to compute the Alexander polynomial of a knot, introduced by the German mathematician Werner Burau in his 1936 paper [8].

Theorem 3.2. *To obtain the Alexander polynomial of a knot K, denoted $\Delta_K(t)$, let f be a braid word of the knot K, f_* be the product of the corresponding matrices from the Burau Representation and n be the number of strands in the braid we use for our braid word. Then*

$$\Delta_K(t) = \frac{1-t}{1-t^n} \det(I - f_*).$$

Example 3.2. *The braid index of the unknot is 1 and there are no terms in its braid word. This implies that $f_* = 0$, which gives us*

$$\Delta_K(t) = \frac{1-t}{1-t} \det(I) = 1.$$

Example 3.3. *By Theorem 2.5, the braid word of the trefoil on a torus $T(2,3)$ is $f = \sigma_1^3 \in B_2$. Since $\psi\sigma_1 = [-t]$, this implies that $f_* = -t^3$ and*

$$\Delta_K(t) = \frac{1-t}{1-t^2}(1-(-t^3)) = \frac{1-t}{1-t^2}(1+t^3) = t^{-1} - 1 + t.$$

Note that multiplying this polynomial by t would lead to the same Alexander polynomial for the trefoil obtained via the defining matrix.

Example 3.4. *Consider the torus knot $K = T(3,5)$. By Theorem 2.5, its braid word is $f = (\sigma_1\sigma_2)^5 \in B_3$. Since*

$$\psi\sigma_1 = \begin{bmatrix} -t & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \psi\sigma_2 = \begin{bmatrix} 1 & 0 \\ t & -t \end{bmatrix},$$

this implies that

$$\psi(\sigma_1\sigma_2) = \begin{bmatrix} 0 & -t \\ t & -t \end{bmatrix} \text{ and } f_* = \psi((\sigma_1\sigma_2)^5) = \begin{bmatrix} -t^5 & t^5 \\ -t^5 & 0 \end{bmatrix}.$$

Then

$$\det(I - f_*) = \begin{vmatrix} 1+t^5 & -t^5 \\ t^5 & 1 \end{vmatrix} = 1 + t^5 + t^{10},$$

and, finally,

$$\Delta_K(t) = \frac{1-t}{1-t^3}(1 + t^5 + t^{10}) = 1 - t + t^2 - t^3 + t^4 - t^5 + t^6 - t^7 + t^8.$$

In Chapter 4, we aim to generalise these examples.

4 Alexander Polynomial of Torus Knots

This chapter is devoted to the investigation of the Alexander polynomial of torus knots, and comparison of torus knots through their corresponding Alexander polynomial. We suppose that the knots $T(p,q)$ are non-trivial, for which $p,q \neq 1$. Note that p and q must be co-prime by Theorem 2.4.

4.1 Via Defining Matrix

We attempt to compute the Alexander polynomial of torus knots via Alexander's definition introduced in Section 3.1.

Theorem 4.1. *Let $K = T(n,2)$ be a torus knot, where n is a positive integer. Then*

$$\Delta_K(t) = \sum_{i=0}^{n-1} (-1)^i x^i.$$

Proof. By Theorem 2.5, the braid word of K must be $\sigma_1^n \in B_2$. We will apply Alexander's method through the braid diagram of σ_1^n. Denote by $N_1, \ldots, N_{n-1}$ the $n-1$ nodes in the diagram, and consider the direction along the braid from the upper rod towards the lower rod. Now denote by R_1 the region on the left side of the braid, and by $R_2, R_3, \ldots, R_{n+1}$ the regions along the braid, starting from the region closest to the upper rod. Finally, denote by R_{n+2} the region on the right side of the braid. Nevertheless, in order to form the knot, we still need to attach the two rods to each other. Upon doing so, the regions R_2 and R_{n+1} will coincide. We thus disregard R_{n+1} as a column in M_K. At each node N_i, we place two dots on the left side of the under-crossing, on the regions R_{i+1} and R_{n+2}. The other two regions around this node are R_1 and R_{i+2}. This gives us the equation

$$R_1 + xR_{i+1} + R_{i+2} + xR_{n+2} = 0$$

at node N_i. For $i = \overline{1, n-1}$, we obtain

$$M_K = \begin{bmatrix} 1 & x & 1 & 0 & \cdots & 0 & x \\ 1 & 0 & x & 1 & \cdots & 0 & x \\ 1 & 0 & 0 & x & \cdots & 0 & x \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 & x \\ 1 & 1 & 0 & 0 & \cdots & x & x \end{bmatrix}.$$

Since R_2 and R_{n+2} are adjacent regions, we write

$$M_K[R_2, R_{n+2}] = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & x & 1 & \cdots & 0 \\ 1 & 0 & x & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & x \end{bmatrix}.$$

This matrix, which we denote by M_n, has size $(n-1) \times (n-1)$. For any positive integer k such that $k \geq 3$, we generalise this matrix to M_k, with size $(k-1) \times (k-1)$. That is, the first column of the matrix M_k only has entries of 1, and each column $i \geq 2$ has the entry 1 on the $(i-1)^{st}$ position and the entry x on the i^{th} position in the column. The rest of the entries in the column are zeroes. Now let $D_k = \det(M_k)$. We aim to determine a recursive formula for D_k. To do so, we apply co-factor expansion along the first row. Note that the only non-zero entries in the first row are the first two entries. For the first entry in the row, we delete the first row and the first column of M_k. The resulting matrix has entries of x on the main diagonal, entries of 1 above each entry on the main diagonal and $0'$s everywhere else. The determinant of this matrix can be determined by multiplying the entries on the main diagonal, obtaining x^{k+1}. For the second entry in the row, we delete the first row and the second column of M_k. The resulting matrix is M_{k-1}. Hence the recursive formula

$$D_k = x^{k-1} - D_{k-1}.$$

By Theorem 2.4, n must be odd so that $2, n$ are co-prime. Then by iteration, we obtain

$$D_k = \sum_{i=3}^{k-1}(-1)^i x^i + D_3 = \sum_{i=0}^{k-1}(-1)^i x^i.$$

This is because

$$D_3 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & x & 1 \\ 1 & 0 & x \end{vmatrix} = \begin{vmatrix} x & 1 \\ 0 & x \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & x \end{vmatrix} = x^2 - x + 1.$$

Hence the proof is complete. □

This method works particularly nicely because $p = 2$, for which we only have the generator σ_1. For this reason, there is a systematic way to label the regions in the braid diagram and track all nodes along the braid. However, it was unclear how to obtain the defining matrix M_K for an arbitrary torus knot $T(p, q)$. Already in the case when $p = 3$, plenty of distinct cases emerge due to the various possible combinations between the powers of σ_1 and σ_2 in the braid words. As a consequence, we devote the next section to compute the Alexander polynomial via the Burau matrices.

4.2 Via Burau Matrices

Consider the torus knot $T(p, q)$, with braid index p. We aim to use Theorem 3.2. To do so, we first need to determine f_*, the product of the corresponding matrices from the Burau representation in the braid word of $T(p, q)$. By Theorem 2.5, the braid word of $T(p, q)$ is $(\sigma_1\sigma_2 \dots \sigma_{p-1})^q$. Thus, we first compute the matrix $\psi(\sigma_1\sigma_2 \dots \sigma_{p-1})$, and then raise it to the power of q. Finally, we will compute one special case of the determinant $\det(I - f_*)$ to compute the Alexander polynomial of another type of torus knots. In this section, we suppose that $p > q$.

4.2.1 Computing the Braid Word Product

We begin with a linear algebra remark. Essentially, what the following lemma states is that, if the block sizes correspond, then block matrix multiplication works in the same way as 'regular' matrix multiplication. As before, empty spots in the matrix will correspond to zero entries.

Lemma 4.2 (Block Matrix Multiplication). *Let A and B be $m \times n$ and $n \times p$ matrices, respectively, written in block form:*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where A_{ij} is $m_i \times n_j$ and B_{ij} is $n_i \times p_j$. Also, $m = m_1 + m_2$, $n = n_1 + n_2$ and $p = p_1 + p_2$. Then

$$C = AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}.$$

Proof. Let $A(i,j)$ denote the (i,j) entry of matrix A and $A_{\alpha\beta}(i,j)$ denote the (i,j) entry of matrix $A_{\alpha\beta}$, where $\alpha, \beta \in \{1,2\}$. We define $B(i,j), C(i,j)$ and $B_{\alpha\beta}(i,j)$ similarly. Consider the term $A_{11}B_{11} + A_{12}B_{21}$. Since A_{11} is $m_1 \times n_1$ and B_{11} is $n_1 \times p_1$, then $A_{11}B_{11}$ is $m_1 \times p_1$ and similarly, so is $A_{12}B_{21}$. For this reason, $1 \leq i \leq m_1$ and $1 \leq j \leq p_1$. By the definition of matrix multiplication, we write

$$C(i,j) = \sum_{k=1}^{n} A(i,k)B(k,j) = \sum_{k=1}^{n_1} A(i,k)B(k,j) + \sum_{k=n_1+1}^{n} A(i,k)B(k,j).$$

Now we can rewrite our coefficients,

$$\begin{aligned} C(i,j) &= \sum_{k=1}^{n_1} A_{11}(i,k)B_{11}(k,j) + \sum_{k=1}^{n_2} A_{12}(i,k)B_{21}(k,j) \\ &= (A_{11}B_{11})(i,j) + (A_{12}B_{21})(i,j) = (A_{11}B_{11} + A_{12}B_{21})(i,j). \end{aligned}$$

The proofs for the other three blocks hold analogously.

□

Proposition 4.1. *Let $\sigma_1, \sigma_2, \ldots, \sigma_{p-1}$ be the generators of B_p. Then*

$$\prod_{i=1}^{j} \sigma_i = \begin{bmatrix} & -t & 1 & \\ tI_{j-1} & -t & 1 & \\ & & I_{p-j-1} & \end{bmatrix}, \forall j \leq p-2, j \geq 2,$$

where the $1'$s above I_{p-j-1} appear in a single column, above the first entry of I_{p-j-1}.

Proof. We prove this claim inductively. For the base step, suppose that $j = 2$. Using the reduced Burau representations, we write:

$$\psi_r(\sigma_1\sigma_2) = \begin{bmatrix} -t & 1 & \\ 0 & 1 & \\ & & I_{p-3} \end{bmatrix} \begin{bmatrix} 1 & 0 & \\ t & -t & \\ & & I_{p-3} \end{bmatrix} = \begin{bmatrix} 0 & -t & 1 \\ t & -t & 1 \\ & & I_{p-3} \end{bmatrix},$$

and the base step is complete. For the induction step, suppose that our claim is true for some $j \geq 2, j < p-2$; we wish to prove that it is true for $j+1$ as well. Direct computation and block matrix multiplication give us

$$\prod_{i=1}^{j+1} \sigma_i = (\prod_{i=1}^{j} \sigma_i)\sigma_{j+1} = \begin{bmatrix} & -t & 1 \\ tI_{j-1} & -t & 1 \\ & & I_{p-j-1} \end{bmatrix} \begin{bmatrix} I_j & & & \\ t & -t & 1 & \\ 0 & 0 & 1 & \\ & & & I_{p-j-3} \end{bmatrix}$$

$$= \begin{bmatrix} & -t & 1 & \\ tI_{j-1} & -t & 1 & \\ & 0 & 1 & \\ & & & I_{p-j-2} \end{bmatrix} \begin{bmatrix} I_{j-1} & & & & \\ & 1 & 0 & 0 & \\ & t & -t & 1 & \\ & & & & I_{p-j-2} \end{bmatrix}$$

$$= \begin{bmatrix} & -t & 1 \\ tI_j & -t & 1 \\ & & I_{p-j-2} \end{bmatrix},$$

and the induction step is complete, which completes the proof. □

Corollary 4.2.1. *Let $\sigma_1, \sigma_2, \ldots, \sigma_{p-1}$ be the generators of B_p. Then*

$$A \coloneqq \prod_{i=1}^{p-1} \sigma_i = \begin{bmatrix} & -t \\ tI_{p-2} & -t \end{bmatrix}, \forall p \geq 2,$$

where the last column is entirely filled with $-t$ entries.

Proof. The proof is a result of Proposition 4.1 and the direct multiplication of $\prod_{i=1}^{p-2} \sigma_i$ by σ_{p-1}. □

Proposition 4.2. *If $2 \leq q \leq p-2$ and $A = \prod_{i=1}^{p-1} \sigma_i$, then*

$$A^q = t^q \begin{bmatrix} & -1 & I_{q-1} \\ & -1 & \\ I_{p-q-1} & -1 & \end{bmatrix}.$$

Proof. We prove this claim inductively and by direct computation. For the base case $q = 2$, we write:

$$A^2 = A \cdot A = t^2 \begin{bmatrix} & -1 \\ I_{p-2} & -1 \end{bmatrix} \begin{bmatrix} & -1 \\ I_{p-2} & -1 \end{bmatrix},$$

and a term by term computation, not included in this paper for the sake of brevity, shows that

$$A^2 = \begin{bmatrix} & -1 & I_{2-1} \\ & -1 & \\ I_{p-2-1} & -1 & \end{bmatrix}.$$

For the induction step, suppose that the claim is true for some $2 \leq q < p-2$. We write

$$A^{q+1} = A^q A = \left(t^q \begin{bmatrix} & -1 & I_{q-1} \\ & -1 & \\ I_{p-q-1} & -1 & \end{bmatrix} \right) \left(t \begin{bmatrix} & -1 \\ I_{p-2} & -1 \end{bmatrix} \right),$$

and once more, a term by term computation gives us

$$A^{q+1} = t^{q+1} \begin{bmatrix} & -1 & I_q \\ & -1 & \\ I_{p-q-2} & -1 & \end{bmatrix},$$

and our proof is complete.

□

Corollary 4.2.2.

$$A^{p-1} = t^{p-1} \begin{bmatrix} -1 & I_{p-2} \\ -1 & \end{bmatrix}.$$

Proof. The proof follows from the previous proposition and the following direct computation,

$$A^{p-1} = A^{p-2}A = t^{p-1} \begin{bmatrix} & -1 & I_{p-3} \\ & -1 & \\ 1 & -1 & \end{bmatrix} \begin{bmatrix} & -1 \\ I_{p-2} & -1 \end{bmatrix}.$$

□

Thus, we obtained a concrete formula for A^q, for any arbitrary $q < p$. Perhaps a notable observation here is that computing A^p would give us $t^p I_{p-1}$. This implies that the sequence

$$u_n = \left\{ \frac{1}{t^n} A^n \right\}$$

of matrices respects the conditions $u_{n+p} = u_n$, for all positive integers n.

4.2.2 Computing the Polynomial

Theorem 4.3. *The Alexander polynomial for the knot $K = T(p, p-1)$ is*

$$\Delta_K(t) = \frac{1-t}{1-t^p} \frac{1-t^{p(p-1)}}{1-t^{p-1}}.$$

Proof. Using Theorem 3.2, the claim becomes proving that

$$\det\left(I_{p-1} - f_*\right) = \frac{1 - t^{p(p-1)}}{1 - t^{p-1}},$$

where

$$f_* = A^{p-1} = t^{p-1}\begin{bmatrix} -1 & I_{p-2} \\ -1 & \end{bmatrix}$$

by Corollary 4.2.2. We prove this inductively. For the base case $p = 2$, we write

$$\det\left(I_{p-1} - f_*\right) = \det\left(1 - (-t)\right) = 1 + t = \frac{1 - t^2}{1 - t}.$$

For the induction step, suppose that the claim is true for some $p \geq 2$. That is, we know that

$$\det\left(I_{p-1} - f_*\right) = \begin{vmatrix} 1 + t^{p-1} & -t^{p-1} & 0 & \cdots & 0 & 0 & 0 \\ t^{p-1} & 1 & -t^{p-1} & \cdots & 0 & 0 & 0 \\ & & & \vdots & & & \\ t^{p-1} & 0 & 0 & \cdots & 1 & -t^{p-1} & 0 \\ t^{p-1} & 0 & 0 & \cdots & 0 & 1 & -t^{p-1} \\ t^{p-1} & 0 & 0 & \cdots & 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{1 - t^{p(p-1)}}{1 - t^{p-1}} = 1 + t^{p-1} + \cdots + t^{(p-1)^2}.$$

Now let

$$\det(I_p - f'_*) = y = \begin{vmatrix} 1+t^{p-1} & -t^{p-1} & 0 & \cdots & 0 & 0 & 0 & 0 \\ t^{p-1} & 1 & -t^{p-1} & \cdots & 0 & 0 & 0 & 0 \\ & & & \vdots & & & & \\ t^{p-1} & 0 & 0 & \cdots & 1 & -t^{p-1} & 0 & 0 \\ t^{p-1} & 0 & 0 & \cdots & 0 & 1 & -t^{p-1} & 0 \\ t^{p-1} & 0 & 0 & \cdots & 0 & 0 & 1 & -t^{p-1} \\ t^{p-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{vmatrix},$$

where this square matrix is one unit larger than the previous one, and f'_* is the product of the corresponding matrices in the new braid word. We wish to show that

$$y = \frac{1 - t^{(p+1)(p-1)}}{1 - t^{p-1}} = 1 + t^{p-1} + \cdots + t^{(p-1)^2} + t^{p(p-1)}.$$

If we apply cofactor expansion along the last row, we get

$$y = y_1 + y_2,$$

where

$$y_1 = (-1)^{p-1} t^{p-1} \begin{vmatrix} -t^{p-1} & 0 & \cdots & 0 & 0 & 0 \\ 1 & -t^{p-1} & \cdots & 0 & 0 & 0 \\ & & \vdots & & & \\ 0 & 0 & \cdots & 1 & -t^{p-1} & 0 \\ 0 & 0 & \cdots & 0 & 1 & -t^{p-1} \end{vmatrix},$$

and

$$y_2 = 1 \cdot \begin{vmatrix} 1+t^{p-1} & -t^{p-1} & 0 & \cdots & 0 & 0 \\ t^{p-1} & 1 & -t^{p-1} & \cdots & 0 & 0 \\ & & & \vdots & & \\ t^{p-1} & 0 & 0 & \cdots & 1 & -t^{p-1} \\ t^{p-1} & 0 & 0 & \cdots & 0 & 1 \end{vmatrix}$$

$$= 1 + t^{p-1} + \cdots + t^{(p-1)^2}$$

by the induction step. Let M_{p-1} represent the $(p-1) \times (p-1)$ matrix whose determinant appears in the computation of y_1, and let M_n be the respective $n \times n$ matrix, for positive integers n. That is, the matrix M_n has entries of $-t^{p-1}$ along the main diagonal, and entries of 1 below the main diagonal. Applying co-factor expansion along the first row of the general matrix term M_n gives us

$$|M_n| = (-t^{p-1})|M_{n-1}|.$$

When $n = p-1, p-2, \ldots, 2$, we obtain the following formula by iteration,

$$|M_{p-1}| = (-t^{p-1})^{p-2}|M_1| = (-t^{p-1})^{p-1},$$

which implies that

$$y_1 = (-1)^{p-1}t^{p-1}(-t^{p-1})^{p-1} = t^{p(p-1)}.$$

We thus conclude that

$$y = 1 + t^{p-1} + \cdots + t^{p(p-1)} = \frac{1 - t^{(p+1)(p-1)}}{1 - t^{p-1}}.$$

□

Hence a concrete form for the Alexander polynomial of a torus knot of the form $T(p, p-1)$.

Surprisingly enough, this theorem can be generalized to any torus knot $T(p,q)$, presented by W. Burau in his 1936 paper [8].

Theorem 4.4. *The Alexander polynomial for the knot $K = T(p,q)$ is*

$$\Delta_K(t) = \frac{1-t}{1-t^p}\frac{1-t^{pq}}{1-t^q}.$$

The proof of this theorem can be attempted in a similar way to the proof of Theorem 4.3. Using Theorem 3.2, the claim becomes proving that

$$\det(I_{p-1} - f_*) = \frac{1-t^{pq}}{1-t^q},$$

where

$$f_* = A^q = t^q \begin{bmatrix} & -1 & I_{q-1} \\ & -1 & \\ I_{p-q-1} & -1 & \end{bmatrix}$$

by Proposition 4.2. Nevertheless, the subsequent computation of the determinant becomes rather complex in this case, for reasons discussed in Chapter 6, along with an exciting possible connection between Knot Theory and Linear Algebra.

4.3 Comparing the Polynomials

Sadly, Alexander polynomials do not distinguish all knots. Nonetheless, amazingly enough, it turns out that the Alexander polynomial defines a torus knot uniquely; that is, the Alexander polynomial of two torus knots is the same for the two knots if and only if the knots are equivalent (up to mirror images). This section aims to prove this statement, for which we present the following theorems. Recall from Chapter 3 that the Alexander polynomial of any knot is the same up to multiplication by $\pm t^n$, for integers n.

Theorem 4.5. *The only torus knot $T(p,q)$ with its Alexander polynomial equal to 1 is the trivial torus knot.*

Proof. Assume by contradiction that there exists a torus knot $T(p, q)$ that is not the unknot (so $p, q > 1$) and for which its Alexander polynomial equals 1. Then $\Delta_K(t) = \pm t^n$, for some integer n. By Theorem 4.4, we have

$$\Delta_K(t) = \frac{(1-t)(1-t^{pq})}{(1-t^p)(1-t^q)},$$

which is a polynomial of degree $1 + pq - p - q = (p-1)(q-1)$. This implies that $n = (p-1)(q-1)$, and by substituting this, we obtain

$$(1-t)(1-t^{pq}) = \pm t^{(p-1)(q-1)}(1-t^p)(1-t^q),$$

which can be extended to the equation

$$1 - t - t^{pq} + t^{pq+1} = \pm t^{(p-1)(q-1)} \mp t^{(p-1)(q-1)+p} \mp t^{(p-1)(q-1)+q} \pm t^{(p-1)(q-1)+p+q}$$

Because of the constant term 1, at least one power of t has its exponent equal to zero. But since $(p-1)(q-1)$ is the smallest such exponent (excluding the term t) and $p, q > 1$, this is impossible. Hence a contradiction, and our proof is complete. □

Corollary 4.5.1. *Every non-trivial torus knot is distinct from the unknot.*

Torus knots are a special type of knots in the sense that the only torus knot whose Alexander polynomial equals 1 is the trivial torus knot. The same result does not hold for knots in general. The Kinoshita-Terasaka knot is an example of a non-trivial knot with Alexander polynomial 1.

We conclude this section with a substantial result in the field of Torus Knot Theory, respectively that the only torus knot equivalent to any given torus knot $T(p, q)$ is $T(q, p)$.

Theorem 4.6. *For $p < q$ and $r < s$, let $K = T(p, q)$ and $L = T(r, s)$ be torus knots with equal Alexander polynomials, up to multiplication of $\pm t^n$. Then $p = r$ and $q = s$.*

Proof. We wish to exclude the trivial torus knot from the comparison, and by Theorem 4.5, this implies that $p, q, r, s > 1$. Suppose that $\Delta_K(t) = \pm t^n \Delta_L(t)$, for some integer n, and without loss of generality assume that n is not negative. The degree of the polynomial on the left hand side is $(p-1)(q-1)$ and the degree of the polynomial on the right hand side is $n+(r-1)(s-1)$, which implies that

$$(p-1)(q-1) = n + (r-1)(s-1). \tag{4.1}$$

After applying Theorem 4.4 and reducing the term $1-t$ from both sides, we obtain the equation

$$\frac{1-t^{pq}}{(1-t^p)(1-t^q)} = \pm t^n \frac{1-t^{rs}}{(1-t^r)(1-t^s)},$$

which leads to

$$(1-t^{pq})(1-t^r-t^s+t^{r+s}) = (\pm t^n \mp t^{n+rs})(1-t^p-t^q+t^{p+q}). \tag{4.2}$$

Once again, the constant term 1 on the left hand side implies that at least one power of t must have a null exponent. But since all exponents except n are surely strictly positive, then n must be zero, and this also determines the signs of the right hand side of the equation. After expanding the brackets and reducing 1 from both sides of Equation 4.2, we obtain

$$-t^r - t^s + t^{r+s} - t^{pq} + t^{r+pq} + t^{s+pq} - t^{r+s+pq} = -t^p - t^q + t^{p+q} - t^{rs} + t^{p+rs} + t^{q+rs} - t^{p+q+rs}$$

Recall that $p<q$ and $r<s$. Then the smallest exponent on the left hand side corresponds to the term $-t^r$ and the smallest exponent on the right hand side corresponds to the term $-t^p$, which implies that $p=r$. Substituting this and $n=0$ into Equation 4.1, we get that $(p-1)(q-1) = (p-1)(s-1)$. We conclude that $q=s$, and our proof is complete. □

5 Alexander Polynomial of Closed 3-Braids

The aim of this section is to determine if non-trivial closed 3-braids could ever have a trivial Alexander polynomial, and to find a closed form for the Alexander polynomials of the braids in B_3. As a brief mention of the case of braids living in B_2, any such closed 2-braid K must have its braid word represented by σ_1^m. If m is even, then the closed 2-braid would simply unravel into the unknot, and if m is odd, say $m = 2n+1$, then f_* in Theorem 3.2 will equal $(-t)^{2n+1}$, in which case

$$\Delta_K(t) = \frac{1-t}{1-t^2}(1+t^{2n+1}) = \sum_{i=0}^{2n}(-t)^i.$$

This polynomial can never equal a monomial of the form $\pm t^p$ for some integer p, which implies that there is no closed 2-braid with braid word of the form σ_1^{2n+1} that has trivial Alexander polynomial. In particular, if $n = 1$, we obtain that the trefoil and the unknot are not equivalent knots, which is proof to a pillar statement in Knot Theory, namely that there exist at least two non-equivalent knots. Turning our focus back to those braids in B_3, some interesting patterns arise along the way, such as the fact that braids with two terms have Alexander polynomials with consecutive exponents only, which is not necessarily the case for braids with at least four terms in their words. By 'term', we refer to some power of a generator. A comfort of working in B_3 comes from only having two generators, σ_1 and σ_2, which means that any braid word with at least three terms must have alternating terms. Moreover, since there are only two generators, the braid word can either start with a power of σ_1 or a power of σ_2. We show first that we can always guarantee that the braid word begins with some power of σ_1.

Lemma 5.1. *Let K be a braid in B_3 whose braid word begins with some power of σ_2. Then there exists a braid K' such that $\Delta_K(t) = \Delta_{K'}(t)$ and the braid word of K' begins with some power of σ_1.*

Proof. Let σ_2^n be the first term in the braid word of K, where n is a non-zero integer, and without loss of generality, suppose that n is positive. Then the braid word of K begins with σ_2, and we can slide the upper rod in the braid representation of K down its first operation σ_2. The braid word of this new braid K_1 begins with σ_2^{n-1} and the remaining σ_2

was moved to the end of the word. We repeat this operation and at each step i, obtain a knot K_i equivalent to K (it is actually the same closed braid representation) whose braid word begins with σ_2^{n-i}. Now let $K' = K_n$. Then the braid word of K' does not begin with σ_2, which implies that it must begin with some power of σ_1. Moreover, K' is equivalent to K, which implies that $\Delta_K(t) = \Delta_{K'}(t)$, and our proof is complete. □

We can also give an entirely algebraic proof of Lemma 5.1 using the following Theorem:

Theorem 5.2 (Sylvester's Determinant Identity)**.** *If A and B are matrices of sizes $p \times q$ and $q \times p$, respectively, then*

$$\det(I_p - AB) = \det(I_q - BA),$$

where I_p and I_q represent the $p \times p$ and $q \times q$ identity matrices.

Alternative Proof of Lemma 5.1. Let f_* represent the product of the corresponding matrices from the Burau representation for K, and suppose that the first term in f_* is some power of σ_2. Re-write $f_* = AB$, where $A = \sigma_2^n$ and B is the rest of the braid word beginning with some power of σ_1. Then by Theorem 3.2 and Sylvester's Identity we obtain that the closed braid K' whose braid word is represented by BA has its Alexander polynomial equal to the Alexander polynomial of K. □

As a consequence of this lemma, we only investigate those braids in B_3 whose braid word begins with some power of σ_1.

Lemma 5.3. *Let K be a braid in B_3 that has an odd number of terms in its braid word. Then there exists a braid K' such that $\Delta_K(t) = \Delta_{K'}(t)$ and the braid word of K' has an even number of terms.*

Proof. By Lemma 5.1, the braid word of K begins with some power of σ_1, say σ_1^m, and since it has an odd number of terms, it must also end with some power of σ_1, say σ_1^n. We apply the same process described in the proof of Lemma 5.1 to obtain a sequence of braids equivalent to K (and which are in fact the same braid representations). Using the same terminology, let $K' = K_m$. Then K' is equivalent to K, which implies that $\Delta_K(t) = \Delta_{K'}(t)$, and K' has one term less in its braid word than K because the term

σ_1^m moved from the beginning of the word to the end of it, thus forming the new term σ_1^{m+n}. Algebraically, this lemma can again be proved with Theorem 3.2 and *Sylvester's Determinant Identity.* □

As a consequence of this lemma, we can limit our investigation to those braids in B_3 that have an even number of terms in their braid word. The following section is devoted to those braids with only two terms in their braid word. The case of four or more is explored in Section 5.2.

5.1 Braid Word: Two Terms

Assume that K is a 3-closed braid whose braid word is represented by $\sigma_1^m \sigma_2^n$, where $m, n \in \mathbb{Z}$, not both zero. We write $u = -t$ for simplicity of notation. Then

$$\psi\sigma_1 = \begin{bmatrix} -t & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} u & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \psi\sigma_2 = \begin{bmatrix} 1 & 0 \\ t & -t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -u & u \end{bmatrix}.$$

Proposition 5.1. *If $m \in \mathbb{Z}$, then*

$$\psi\sigma_1^m = \begin{bmatrix} u^m & \frac{1-u^m}{1-u} \\ 0 & 1 \end{bmatrix}.$$

Proof. We first examine the case when m is a natural number, by induction. The base step $m = 1$ is trivial. For the induction step, suppose the claim to be true for some natural number m. Then

$$\psi\sigma_1^{m+1} = (\psi\sigma_1^m)(\psi\sigma_1) = \begin{bmatrix} u^m & \frac{1-u^m}{1-u} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} u^{m+1} & u^m + \frac{1-u^m}{1-u} \\ 0 & 1 \end{bmatrix},$$

and the step is concluded by noting that

$$u^m + \frac{1-u^m}{1-u} = \frac{u^m - u^{m+1} + 1 - u^m}{1-u} = \frac{1-u^{m+1}}{1-u}.$$

Now consider the case when m is a negative integer. Note that $|\sigma_1| = u$ and so

$$\psi\sigma_1^{-1} = \frac{1}{u}\begin{bmatrix} 1 & -1 \\ 0 & u \end{bmatrix} = \begin{bmatrix} u^{-1} & -u^{-1} \\ 0 & 1 \end{bmatrix}.$$

Again, we prove the claim by induction; the base step follows from the previous observation, along with the identity

$$\frac{1-u^{-1}}{1-u} = \frac{u(1-u^{-1})}{u(1-u)} = \frac{u-1}{u(1-u)} = -u^{-1}.$$

For the induction step, assume the claim to be true for some $m \in \mathbb{Z}_-$. Then

$$\psi\sigma_1^{m-1} = (\psi\sigma_1^m)(\psi\sigma_1^{-1}) = \begin{bmatrix} u^m & \frac{1-u^m}{1-u} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u^{-1} & -u^{-1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} u^{m-1} & -u^{m-1} + \frac{1-u^m}{1-u} \\ 0 & 1 \end{bmatrix},$$

and the step is concluded after the observation that

$$-u^{m-1} + \frac{1-u^m}{1-u} = \frac{-u^{m-1} + u^m + 1 - u^m}{1-u} = \frac{1-u^{m-1}}{1-u}.$$

□

It is noteworthy to mention here that the term $\frac{1-u^m}{1-u}$ is a polynomial and can be expressed as

$$\frac{1-u^m}{1-u} = \sum_{i=0}^{m-1} u^i$$

if m is a natural number, and

$$\frac{1-u^m}{1-u} = -\sum_{i=1}^{-m} u^{-i}$$

if m is a negative integer.

Proposition 5.2. *If n is an integer, then*

$$\psi\sigma_2^n = \begin{bmatrix} 1 & 0 \\ -u\frac{1-u^n}{1-u} & u^n \end{bmatrix}.$$

Proof. If n is positive, we prove the claim by induction. The base step is trivial, and for the induction step, suppose the claim to be true for some natural number n. Then

$$\psi\sigma_2^{n+1} = (\psi\sigma_2^n)(\psi\sigma_2) = \begin{bmatrix} 1 & 0 \\ -u\frac{1-u^n}{1-u} & u^n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -u & u \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -u\frac{1-u^n}{1-u} - u^{n+1} & u^{n+1} \end{bmatrix},$$

and the induction is complete with the following remark:

$$-u\frac{1-u^n}{1-u} - u^{n+1} = -u\left(\frac{1-u^n}{1-u} + u^n\right)$$

$$= -u\frac{1-u^{n+1}}{1-u}.$$

Now consider the case when n is a negative integer. Note that $|\sigma_2| = u$ and we can write

$$\psi\sigma_2^{-1} = \frac{1}{u}\begin{bmatrix} u & 0 \\ u & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & u^{-1} \end{bmatrix};$$

this completes the base case of our following induction (for negative exponents), since

$$-u\frac{1-u^{-1}}{1-u} = -\frac{u-1}{1-u} = 1.$$

For the induction step, suppose that the claim is true for some negative integer n. Then

$$\psi\sigma_2^{n-1} = (\psi\sigma_2^n)(\psi\sigma_2^{-1}) = \begin{bmatrix} 1 & 0 \\ -u\frac{1-u^n}{1-u} & u^n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & u^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -u\frac{1-u^n}{1-u} + u^n & u^{n-1} \end{bmatrix},$$

and our induction is complete if we note that

$$-u\frac{1-u^n}{1-u}+u^n=-u\left(\frac{1-u^n}{1-u}-u^{n-1}\right)$$

$$=-u\frac{1-u^{n-1}}{1-u}.$$

□

Again, the term $-u\frac{1-u^n}{1-u}$ is a polynomial and can be expressed as

$$-u\frac{1-u^n}{1-u}=-\sum_{i=1}^{n}u^i$$

if n is a natural number, and

$$-u\frac{1-u^n}{1-u}=\sum_{i=0}^{-(m+1)}u^{-i}$$

if n is a negative integer.

Proposition 5.3. *If K is a closed 3-braid with braid word $\sigma_1^m\sigma_2^n$, then*

$$\Delta_K(u)=\frac{1-u^m}{1-u}\frac{1-u^n}{1-u}.$$

Proof. With the aid of Proposition 5.1 and Proposition 5.2, we can compute $\psi\sigma_1^m\sigma_2^n$,

$$\psi\sigma_1^m\sigma_2^n=\begin{bmatrix}u^m & \frac{1-u^m}{1-u}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -u\frac{1-u^n}{1-u} & u^n\end{bmatrix}=\begin{bmatrix}u^m-u\frac{1-u^m}{1-u^n} & u^n\frac{1-u^m}{1-u}\\ -u\frac{1-u^n}{1-u} & u^n\end{bmatrix},$$

which implies that

$$|I - \sigma_1^m \sigma_2^n| = \left(1 - u^m + u\frac{1-u^m}{1-u}\frac{1-u^n}{1-u}\right)(1-u^n) + u^{n+1}\frac{1-u^m}{1-u}\frac{1-u^n}{1-u} =$$
$$= 1 - u^n - u^m + u^{n+m} + u\frac{1-u^m}{1-u}\frac{1-u^n}{1-u} - u^{n+1}\frac{1-u^m}{1-u}\frac{1-u^n}{1-u} +$$
$$+ u^{n+1}\frac{1-u^m}{1-u}\frac{1-u^n}{1-u} = (1-u^n)(1-u^m) + \frac{u}{(1-u)^2}(1-u^n)(1-u^m) =$$
$$= (1-u^m)(1-u^n)\left(1 + \frac{u}{(1-u)^2}\right) = \frac{(1-u^m)(1-u^n)}{(1-u)(1-u)}(1 - 2u + u^2 + u) =$$
$$= \frac{1-u^m}{1-u}\frac{1-u^n}{1-u}(1 - u + u^2).$$

If we apply Theorem 3.2, we get

$$\Delta_K(u) = \frac{1-t}{1-t^3}\det(I - f_*) = \frac{1+u}{1+u^3}(1-u+u^2)\frac{1-u^m}{1-u}\frac{1-u^n}{1-u},$$

and by the identity $1 + u^3 = (1+u)(1-u+u^2)$, we conclude with our final result

$$\boxed{\Delta_K(u) = \frac{1-u^m}{1-u}\frac{1-u^n}{1-u}}.$$

□

Theorem 5.4. *If K is a closed 3-braid with braid word $\sigma_1^m \sigma_2^n$, then K has a trivial Alexander polynomial if and only if K is the unknot.*

Proof. Since the Alexander polynomial of a braid is unique up to a multiplication of $\pm t^p$, where p is some integer, the question becomes whether there exist some integers m, n, not both zero, such that

$$\Delta_K(u) = \pm(-u)^p,$$

for some integer p, where $K = \sigma_1^m \sigma_2^n$. In $\Delta_K(u)$, the degree of the numerator (a polynomial) is $m+n$ and the degree of the denominator is 2, which implies that p must equal $m+n-2$. Also note that we can simply write $\pm(-u)^p$ as $\pm u^p$. Then the equation $\Delta_K(u) = \pm u^p$ can be written as

$$\frac{1-u^m}{1-u}\frac{1-u^n}{1-u} = \pm u^{m+n-2} \Rightarrow (1-u^m)(1-u^n) = \pm u^{m+n-2}(1-2u+u^2) \Rightarrow$$
$$\Rightarrow 1-u^m-u^n+u^{m+n} = \pm u^{m+n-2} \mp 2u^{m+n-1} \pm u^{m+n},$$

and we examine each of the two resulting equations separately.

1. $1-u^m-u^n+u^{m+n} = u^{m+n-2} - 2u^{m+n-1} + u^{m+n}$, which implies that

$$u^{m+n-2} - 2u^{m+n-1} + u^m + u^n - 1 = 0.$$

We must have $m+n-2=0$ or $m+n-1=0$ or $m=0$ or $n=0$, and we evaluate each case separately.

   (a) $m+n=2 \Rightarrow 1-2u+u^m+u^n-1=0 \Rightarrow u^m+u^n-2u=0$, with the unique solution $(m,n)=(1,1)$, which corresponds to the unknot.

   (b) $m+n=1 \Rightarrow u^{-1}+u^m+u^n-3=0$, with no solutions.

   (c) $m=0 \Rightarrow u^{n-2}-2u^{n-1}+u^n=0 \Rightarrow u^{n-2}(1-u)^2=0$, with no solutions.

   (d) The case $n=0$ is equivalent to $m=0$ because the equation in question is symmetric in m and n.

2. $1-u^m-u^n+u^{m+n} = -u^{m+n-2} + 2u^{m+n-1} - u^{m+n}$, which gives us

$$2u^{m+n} - 2u^{m+n-1} + u^{m+n-2} - u^m - u^n + 1 = 0.$$

Again, we must have $m+n=0$ or $m+n-1=0$ or $m+n-2=0$ or $m=0$ or $n=0$, and we evaluate each case separately.

   (a) $m+n=0 \Rightarrow n=-m \Rightarrow -2u^{-1}+u^{-2}-u^m-u^{-m}+3=0$, with no solutions.

   (b) $m+n=1 \Rightarrow 2u+u^{-1}-u^m-u^n-1=0$, with no solutions.

   (c) $m+n=2 \Rightarrow 2u^2-2u-u^m-u^n+2=0$, with no solutions.

   (d) $m=0 \Rightarrow u^n-2u^{n-1}+u^{n-2}=0 \Rightarrow u^{n-2}(1-u)^2=0$, with no solutions.

(e) The case $n = 0$ is equivalent to the case when $m = 0$ by symmetry.

□

Hence in B_3, for 3-closed braids with braid words of the form $\sigma_1^m \sigma_2^n$, the only case in which the Alexander polynomial of K is equivalent to that of the unknot is if K is itself the unknot.

5.2 Braid Words: Four Terms or More

Suppose that K is a 3-closed braid whose braid word is represented by $\sigma_1^m \sigma_2^n \sigma_1^p \sigma_2^q$, where m, n, p, q are non-zero integers.

Proposition 5.4. *If K is a closed 3-braid with braid word $\sigma_1^m \sigma_2^n \sigma_1^p \sigma_2^q$, then*

$$\Delta_K(u) = \frac{M - Nu + Pu^2}{(1-u)^4},$$

where

$$M = (1 - u^{m+p})(1 - u^{n+q}),$$

$$N = 3 - u^m - u^n - u^p - u^q + u^{m+n} - u^{m+p} + u^{m+q} + u^{n+p} - u^{n+q} + u^{p+q} - u^{m+n+p} - u^{m+n+q} - \\ - u^{m+p+q} - u^{n+p+q} + 3u^{m+n+p+q}$$

and

$$P = (1 - u^{m+p})(1 - u^{n+q}).$$

Proof. From the previous case, we know that

$$\psi \sigma_1^m \sigma_2^n = \begin{bmatrix} u^m - u\frac{1-u^m}{1-u^n} & u^n \frac{1-u^m}{1-u} \\ -u\frac{1-u^n}{1-u} & u^n \end{bmatrix}$$

and

$$\psi\sigma_1^p\sigma_2^q = \begin{bmatrix} u^p - u\frac{1-u^p}{1-u^q} & u^p\frac{1-u^p}{1-u} \\ -u\frac{1-u^q}{1-u} & u^q \end{bmatrix}.$$

Therefore, we can multiply these matrices and obtain

$$\psi\sigma_1^m\sigma_2^n\sigma_1^p\sigma_2^q = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where

$$A = u^{m+p} - u^{m+1}\frac{1-u^p}{1-u}\frac{1-u^q}{1-u} - u^{p+1}\frac{1-u^m}{1-u}\frac{1-u^n}{1-u} + \\ + u^2\frac{1-u^m}{1-u}\frac{1-u^n}{1-u}\frac{1-u^p}{1-u}\frac{1-u^q}{1-u} - u^{n+1}\frac{1-u^m}{1-u}\frac{1-u^q}{1-u},$$

$$B = u^{m+p}\frac{1-u^q}{1-u} - u^{p+1}\frac{1-u^m}{1-u}\frac{1-u^n}{1-u}\frac{1-u^q}{1-u} + u^{n+q}\frac{1-u^m}{1-u},$$

$$C = -u^{p+1}\frac{1-u^n}{1-u} + u^2\frac{1-u^n}{1-u}\frac{1-u^p}{1-u}\frac{1-u^q}{1-u}$$

and

$$D = -u^{p+1}\frac{1-u^n}{1-u}\frac{1-u^q}{1-u} + u^{n+q}.$$

From here, we can introduce the notation $u_x = \frac{1-u^x}{1-u}$, for any x, to simplify our entries. We can thus re-write the entries of f_* as follows:

$$A = u^{m+p} - u^{m+1}u_pu_q - u^{p+1}u_mu_n + u^2u_mu_nu_pu_q - u^{n+1}u_mu_q,$$

$$B = u^{m+p}u_q - u^{p+1}u_m u_n u_q + u^{n+q}u_m,$$

$$C = -u^{p+1}u_n + u^2 u_n u_p u_q \text{ and } D = -u^{p+1}u_n u_q + u^{n+q}.$$

□

Already this early in our investigation, the computations become infeasible by hand, and we turn to Mathematica, which gives us the entries described in the claim of the theorem. It is perhaps noteworthy to mention here that $M = P$, and this also holds true for the equivalent coefficients in the case when K is a closed 3-braid with braid word represented by $\sigma_1^m \sigma_2^n \sigma_1^p \sigma_2^q \sigma_1^r \sigma_2^z$, in which case Mathematica outputs that

$$\Delta_K(u) = \frac{M - Nu + Pu^2 - Qu^3 + Ru^4}{(1-u)^6},$$

where

$$M = R = (1 - u^{m+p+r})(1 - u^{n+q+z}).$$

Theorem 5.5. *If K is a closed 3-braid with braid word $\sigma_1^m \sigma_2^n \sigma_1^p \sigma_2^q$, then K has a trivial Alexander polynomial if and only if K is the unknot.*

Proof. Since the Alexander polynomial of a braid is unique up to a multiplication of $\pm t^r$, where r is some integer, the question becomes whether there exist some non-zero integers m, n, p, q such that

$$\Delta_K(u) = \frac{M - Nu + Pu^2}{(1-u)^4} = \pm(-u)^r,$$

for some integer r. Assume by contradiction that such integers exist. In $\Delta_K(u)$, the degree of the numerator (a polynomial) is $m + n + p + q$ and the degree of the denominator is 2, which implies that r must equal $m + n + p + q - 2$. Denote

$$LHS = M - Nu + Pu^2 \text{ and } RHS = (1-u)^4(-u)^{m+n+p+q-2}.$$

Then either $LHS = RHS$ or $LHS = -RHS$, and we investigate the former case first. After expanding the equation and eliminating the brackets, the term $-3u$ emerges. Because of this, at least three other exponents must equal one. Divide now the exponents into three sets:

$$S_1 = \{m+n+p+q-1, m+n+p+q, m+n+p+q+1, m+n+p+q+2\}, S_2 = \{m+p, n+q\}$$

and

$$S_3 = \{m+n+p, m+n+q, m+p+q, n+p+q\}.$$

Note that S_1 contains consecutive elements, which means that at most one among them can be zero. Suppose first that no elements of S_1 are zero. Further note that if $m+p=0$, then we cannot have $m+n+p=0$ or $m+p+q=0$ (m, n, p, q are non-zero) and similarly, if $n+q=0$, then we cannot have $m+n+q=0$ or $n+p+q=0$. This implies that either $m+p=0$ or $n+q=0$ or neither equals zero, and we evaluate each case separately.

If $m+p=0$, since $n+q$ cannot be zero, we must have $m+n+q=0$ and $n+p+q=0$, which together imply that $m=p$, and this is impossible. The case $n+q=0$ follows analogously, and we obtain a contradiction. Now suppose that neither $m+p$ nor $n+q$ equals zero. Then three elements of S_3 must be zero. Suppose that $m+n+p=0, m+n+q=0$ and $m+p+q=0$ (the other cases follow similarly). The first two equations imply that $p=q$ and the last two equations imply that $n=p$, and we get that $n=p=q$ and also that $m=-2n$. Because of the terms 1 and u^2 that emerge on the left hand side of the equation, we need at least one more exponent to equal zero and another to equal two in order to cancel out these two terms. Again, we divide the exponents into three sets:

$$T_1 = \{m+p, n+q, m+p+2, n+q+2\}, T_2 = \{m+n+1, m+q+1, n+p+1, p+q+1\}$$

and

$$T_3 = \{m+n+p+q-2, m+n+p+q, m+n+p+q+1, m+n+p+q+2\}.$$

Taking our context into account as well, the elements $-n, -n+1, -n+2, n-2, n, n+1, n+2, 2n, 2n+1$ and $2n+2$ remain in the three sets altogether. At least one element equals zero; however, n is non-zero, which implies that n must equal $-2, -1, 1$ or 2. If $n = -1$ or $n = 2$, then two exponents are zero, which cannot cancel out the term 1. If $n = 1$, then the braid word of K becomes $\sigma_1^{-2}\sigma_2\sigma_1\sigma_2$, which can be re-written as $\sigma_1^{-2}\sigma_1\sigma_2\sigma_1$, or simply $\sigma_1^{-1}\sigma_2\sigma_1$ (this follows from the second rule in Definition 2.7), and this can further be re-written as a braid with only two terms in its braid word (as a consequence of the proof of Lemma 5.3). Finally, if $n = -2$, then two exponents equal two, which cannot happen, and we reached a contradiction.

Suppose now that one element of S_1 is zero and that two elements in S_2 and S_3 combined are also zero. Then $m+n+p+q$ can only equal $-2, -1, 0$ or 1. Now there is a chance that $m+p$ and $n+q$ are both zero, but in this case, two exponents in the T sets equal zero, and this is impossible. Therefore, we distinguish two remaining cases.

In the first case, either $m+p = 0$ or $n+q = 0$, and without loss of generality suppose that $m+p = 0$. We must have one last element equal zero from S_3, and this can only be $m+n+q$ or $n+p+q$. Suppose that $m+n+q = 0$, for the other case follows analogously. Since $m+n+p+q = n+q$, then $m+n+p+q$ cannot equal zero, and it can only be $-2, -1$ or 1. If it equals -2 or -1, then an extra exponent in the T sets will equal zero, and this is a contradiction. Then $m+n+p+q = 1$, which implies that an extra exponent in the T sets equals two, impossible.

In the second case, neither $m+p = 0$ nor $n+q = 0$. Two elements of S_3 must be zero, and without loss of generality suppose that $m+n+p = m+n+q = 0$, which implies that

$p = q$ and $m + n + p + q = p \neq 0$. Thus, we examine the cases when $p = -2, -1$ or 1. If $p = q = -2$, then from our T sets it follows that one of the following exponents equals two: $m+p, n+q, m+n+1, m+q+1, n+p+1, p+q+1$. If $m+p = 2$, then $m = 4$ and since $m+n+p = 0$, we obtain $n = -2$, which leads to the solution $(m, n, p, q) = (4, -2, -2, -2)$. This, however, fails when substituted back into the original equation because the left hand side would contain a unique power of 5, whereas the right hand side would not. The case when $n + q = 2$ follows similarly. If $m + n + 1 = 2$, then $m + n = 1$ and $m + n + p = -1$, which is a contradiction. If $m + q + 1 = 2$, then $m + p = 1$, which implies that $m = 3$, and since $m+n+p = 0$, then $n = -1$. This gives us the solution $(m, n, p, q) = (3, -1, -2, -2)$, which fails when substituted back into the initial equation because the left hand side would contain a unique power of 4, whereas the right hand side would not. The case when $n + p + 1 = 0$ follows similarly. Finally, note that $p + q + 1$ cannot equal 2 because $p = q$. The cases when $p = q = -1$ and $p = q = 1$ follow with a similar analysis, and this concludes our proof for this general case when $LHS = RHS$. The case $LHS = -RHS$ follows analogously, and our proof is complete. □

Hence in B_3, for 3-closed braids K that have braid words of the form $\sigma_1^m \sigma_2^n \sigma_1^p \sigma_2^q$, the only case in which the Alexander polynomial of K is equivalent to that of the unknot is if K is itself the unknot. If K contains six terms or more in its braid word, the computation becomes infeasible by hand for both the product of the corresponding matrices from the Burau Representation and especially the determinant that arises in Theorem 3.2. However, with the aid of Mathematica we can note that a pattern does seem to emerge, at least for the first few cases, which motivates the following conjecture.

Conjecture 5.1. *Suppose that K is a closed 3-braid with braid word represented by*

$$\prod_{i=1}^{l} (\sigma_1^{m_i} \sigma_2^{n_i}).$$

Then the Alexander polynomial of K can be written in the form

$$\Delta_K(u) = \frac{\sum_{i=0}^{2l-2} (-1)^i u^i A_i}{(1-u)^{2l}},$$

where

$$A_0 = A_{2l-2} = \left(1 - u^{\sum_{i=1}^{l} m_i}\right)\left(1 - u^{\sum_{j=1}^{l} n_j}\right).$$

Overall, this process hints at a different approach to proving the following theorem from Stoimenow [1].

Theorem 5.6 ([1])**.** *Suppose that K is a closed 3-braid and that U is the unknot. Then* $\Delta_K(u) \neq \Delta_U(u)$.

6 Future Directions and Open Questions

Certainly a challenge of this project was narrowing down the research, due to the exuberant number of open questions in the field, and the complexity with which each one of them unravels. For this reason, it is only natural to devote an entire section to open questions and future directions that can lead up from this work. The most straightforward open question of this research remains a follow-up to the discussion in Section 4.2.2 regarding the Alexander polynomial of a torus knot in general; that is, a proof for Theorem 4.4. The main difficulty of proving this theorem computationally like we did in Theorem 4.3 arises from the computation of the determinant $\det(I - f_*)$. The complexity of the situation is caused by the full column of -1 entries that appears on column $p - q$ in the matrix f_*. In this case, we consider $p > q$. Because q can have any value between 2 and $p - 2$, when subtracting f_* from the identity matrix, keeping track of the position of all entries becomes tedious, and computing the determinant becomes tricky without use of technology. The additional fact that p and q are relatively prime also restricts our use of induction, an essential tool in the proof of Theorem 4.3. Surprisingly enough, it does matter algebraically that p and q are relatively prime, for otherwise induction could still have been used. Take the torus link $L = T(6, 3)$, for example. By Theorem 2.3, the torus link L is equivalent to the torus link $T(3, 6)$, which has its braid word represented by $(\sigma_1\sigma_2)^6$ by Theorem 2.5. The product of the corresponding matrices from the Burau Representation can be obtained by direct computation,

$$f_* = \begin{bmatrix} t^6 & 0 \\ 0 & t^6 \end{bmatrix} = t^6 I_2 \Rightarrow I_2 - f_* = (1 - t^6) I_2.$$

This leads to the determinant of $I_2 - f_*$ being $(1 - t^6)^2 = 1 - 2t^6 + t^{12}$. However, the determinant would need to equal (if p and q were relatively prime)

$$\frac{1 - t^{pq}}{1 - t^q} = \frac{1 - t^{18}}{1 - t^6} = 1 + t^6 + t^{12},$$

and we obtain two clearly distinct polynomials. Consequently, it follows that the co-primeness of p and q indeed affects the algebraic computations of the determinant. This connection would definitely be an exciting topic for further studies, and it may also give insights for the actual computation of the determinant.

There are numerous exploratory paths surrounding this topic. Instead of comparing the Alexander polynomials of various knots with that of the unknot, and with each other, we could derive the distinct types of polynomials that emerge as Alexander polynomials, and tackle the question whether every polynomial in $\mathbb{R}[t, t^{-1}]$ is an Alexander polynomial for some knot. For example, the exponents of the Alexander polynomial of a closed 3-braid with braid word of the form $\sigma_1^m \sigma_2^n$ can be consecutive integers only, but that is not the case for braid words of the form $\sigma_1^m \sigma_2^n \sigma_1^p \sigma_2^q$, where the consecutive exponents may skip a number. The patterns in the exponents become even more captivating for closed 3-braids with at least six terms in their braid word, and it must be the same case for general closed n-braids.

An immediate question that remains open in Chapter 5 would be whether the conjecture stated in the end of the section was true and if it could be used to obtain Stoimenow's Theorem. It is perhaps needless to mention that another future direction of this research would be examining the Alexander polynomials of closed braids living in B_n, for larger integers n. Unsurprisingly, the computations that follow along from the B_3 case become increasingly more complex, and even classifying the braid words becomes tedious, resulting into more cases to consider. Take the succeeding case, for example, with three generators. The only available tool remaining for classifying the braid words is that we can re-write each braid word such that it begins with some power of σ_1. Other than that, the second term in the braid word could be either σ_2 or σ_3, and from there, we have no control over the order of the generators. Another limitation of the methods described in this paper is that comparing the Alexander polynomial of a given knot to even the trivial Alexander polynomial can become so computationally heavy that it requires assistance from computers, and so far no substantial methods for comparing polynomials emerged, other than brute force. One possible way to overlook these computations would be by classifying knots into categories (their braid index could be one classifying aspect), and then studying in each category, the types of Alexander polynomials that emerge, and the patterns they form. Using this information, we could eventually determine which categories may contain knots with trivial Alexander polynomial, and evaluate those separately.

Last, but not least, a similar research can be conducted for links instead of knots. As we mentioned previously, results such as Theorem 2.3 remain valid for links, and they can constitute the starting results of the research. An interesting aspect of this would be that in the case of torus links $T(p, q)$, the positive integers p and q cannot be co-prime, which would make the investigation of the matrices involved as intriguing as the case of torus knots. The results that we derived for general knots would not change, however, because the analysis of 3-closed braids in Chapter 5 did not take into account whether the braid words would generate knots or links. This means that essentially, we are comparing knots and links together; perhaps an improvement of the methods in Chapter 5 would be finding a systematic way to differentiate the braid words of knots and links, which would allow us to separate our investigation.

7 Appendix

Claim. *The matrices given by the Burau representation respect the braid relations that define the braid group B_n algebraically:*

1. $\sigma_i \sigma_j = \sigma_j \sigma_i$ *for* $|i - j| \geq 2$.

2. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ *for* $1 \leq i \leq n-2$.

Proof. We start with the first property, and break down our investigation into cases, which will determine the forms of σ_i and σ_j from the Reduced Burau Representations.

1. If $i = 1$, then $j \geq 3$ and we have the following two subcases:

   (a) If $j = n-1$, we have:

$$\psi_r \sigma_1 = \begin{bmatrix} -t & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{bmatrix} \text{ and } \psi_r \sigma_{n-1} = \begin{bmatrix} I_{n-3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & -t \end{bmatrix}.$$

   We can re-write $\psi_r \sigma_{n-1}$ as

$$\psi_r \sigma_{n-1} = \begin{bmatrix} 1 & 0 & & & \\ 0 & 1 & & & \\ & & I_{n-5} & & \\ & & & 1 & 0 \\ & & & t & -t \end{bmatrix}$$

   and apply block matrix multiplication to write

$$\psi_r (\sigma_1 \sigma_{n-1}) = \begin{bmatrix} -t & 1 & & & \\ 0 & 1 & & & \\ & & I_{n-5} & & \\ & & & 1 & 0 \\ & & & t & -t \end{bmatrix}$$

and

$$\psi_r(\sigma_{n-1}\sigma_1) = \begin{bmatrix} -t & 1 & & & \\ 0 & 1 & & & \\ & & I_{n-5} & & \\ & & & 1 & 0 \\ & & & t & -t \end{bmatrix},$$

which shows that $\psi_r(\sigma_1\sigma_{n-1}) = \psi_r(\sigma_{n-1}\sigma_1)$ and so $\sigma_1\sigma_{n-1} = \sigma_{n-1}\sigma_1$.

(b) If $j \neq n-1$, then $3 \leq j \leq n-2$ and

$$\psi_r\sigma_j = \begin{bmatrix} I_{j-2} & & & & \\ & 1 & 0 & 0 & \\ & t & -t & 1 & \\ & 0 & 0 & 1 & \\ & & & & I_{n-j-2} \end{bmatrix}.$$

If $j \geq 4$, then we re-write this as

$$\psi_r\sigma_j = \begin{bmatrix} 1 & 0 & & & & & \\ 0 & 1 & & & & & \\ & & I_{j-4} & & & & \\ & & & 1 & 0 & 0 & \\ & & & t & -t & 1 & \\ & & & 0 & 0 & 1 & \\ & & & & & & I_{n-j-2} \end{bmatrix}$$

and apply block matrix multiplication to get

$$\psi_r(\sigma_1\sigma_j) = \begin{bmatrix} -t & 1 & & & & & \\ 0 & 1 & & & & & \\ & & I_{j-4} & & & & \\ & & & 1 & 0 & 0 & \\ & & & t & -t & 1 & \\ & & & 0 & 0 & 1 & \\ & & & & & & I_{n-j-2} \end{bmatrix}$$

and

$$\psi_r(\sigma_j\sigma_1) = \begin{bmatrix} -t & 1 & & & & & \\ 0 & 1 & & & & & \\ & & I_{j-4} & & & & \\ & & & 1 & 0 & 0 & \\ & & & t & -t & 1 & \\ & & & 0 & 0 & 1 & \\ & & & & & & I_{n-j-2} \end{bmatrix},$$

which shows that $\psi_r(\sigma_1\sigma_j) = \psi_r(\sigma_j\sigma_1)$ and so $\sigma_1\sigma_j = \sigma_j\sigma_1$. If, on the contrary, $j < 4$, then $j = 3$ and

$$\psi_r\sigma_3 = \begin{bmatrix} 1 & 0 & & & \\ 0 & 1 & & & \\ 0 & t & -t & 1 & \\ 0 & 0 & 0 & 1 & \\ & & & & I_{n-5} \end{bmatrix},$$

where note that the entry t below the block matrix I_2 within $\psi_r\sigma_3$ is singular, and by applying block matrix multiplication we get

$$\psi_r(\sigma_1\sigma_3) = \begin{bmatrix} -t & 1 & & & \\ 0 & 1 & & & \\ & t & -t & 1 & \\ & & 0 & 1 & \\ & & & & I_{n-5} \end{bmatrix}$$

and

$$\psi_r(\sigma_3\sigma_1) = \begin{bmatrix} -t & 1 & & & \\ 0 & 1 & & & \\ & t & -t & 1 & \\ & & 0 & 1 & \\ & & & & I_{n-5} \end{bmatrix},$$

where again note the singular t entry. This shows that $\psi_r(\sigma_1\sigma_3) = \psi_r(\sigma_3\sigma_1)$ and so $\sigma_1\sigma_3 = \sigma_3\sigma_1$, which completes this case.

2. If $i \neq 1$, then $2 \leq i \leq n-3$ and we have

$$\psi_r\sigma_i = \begin{bmatrix} I_{i-2} & & & & \\ & 1 & 0 & 0 & \\ & t & -t & 1 & \\ & 0 & 0 & 1 & \\ & & & & I_{n-i-2} \end{bmatrix}.$$

Once more, we examine two subcases.

(a) If $j = n-1$, then

$$\psi_r\sigma_{n-1} = \begin{bmatrix} I_{n-3} & & \\ & 1 & 0 \\ & t & -t \end{bmatrix}.$$

If $i \leq n-4$, then $n-i-2 \geq 2$ and we can re-write $\psi_r\sigma_i$ as following:

$$\psi_r\sigma_i = \begin{bmatrix} I_{i-2} & & & & & & \\ & 1 & 0 & 0 & & & \\ & t & -t & 1 & & & \\ & 0 & 0 & 1 & & & \\ & & & & I_{n-i-4} & & \\ & & & & & 1 & 0 \\ & & & & & 0 & 1 \end{bmatrix}.$$

We apply block matrix multiplication and obtain:

$$\psi_r(\sigma_i\sigma_{n-1}) = \begin{bmatrix} I_{i-2} & & & & & & \\ & 1 & 0 & 0 & & & \\ & t & -t & 1 & & & \\ & 0 & 0 & 1 & & & \\ & & & & I_{n-i-4} & & \\ & & & & & 1 & 0 \\ & & & & & t & -t \end{bmatrix} = \psi_r(\sigma_{n-1}\sigma_i),$$

which gives us $\sigma_i\sigma_{n-1} = \sigma_{n-1}\sigma_i$. If $i > n-4$, then $i = n-3$ and we have

$$\psi_r\sigma_{n-3} = \begin{bmatrix} I_{n-5} & & & & \\ & 1 & 0 & 0 & \\ & t & -t & 1 & \\ & 0 & 0 & 1 & \\ & & & & 1 \end{bmatrix},$$

which we can re-write as

$$\psi_r\sigma_{n-3} = \begin{bmatrix} I_{n-5} & & & & \\ & 1 & 0 & & \\ & t & -t & 1 & \\ & & & 1 & 0 \\ & & & 0 & 1 \end{bmatrix},$$

where note that the entry 1 above the block matrix I_2 within $\psi_r\sigma_{n-3}$ is singular, to obtain that

$$\psi_r(\sigma_{n-3}\sigma_{n-1}) = \begin{bmatrix} I_{n-5} & & & & \\ & 1 & 0 & & \\ & t & -t & 1 & \\ & & & 1 & 0 \\ & & & t & -t \end{bmatrix} = \psi_r(\sigma_{n-1}\sigma_{n-3}),$$

which implies that $\sigma_{n-3}\sigma_{n-1} = \sigma_{n-1}\sigma_{n-3}$. Once more, note the singular 1 entry.

(b) If $j \neq n-1$, then $i+2 \leq j \leq n-2$ and we can write

$$\psi_r\sigma_j = \begin{bmatrix} I_{j-2} & & & & \\ & 1 & 0 & 0 & \\ & t & -t & 1 & \\ & 0 & 0 & 1 & \\ & & & & I_{n-j-2} \end{bmatrix},$$

and we can further re-write this matrix as

$$\psi_r\sigma_j = \begin{bmatrix} I_{i-2} & & & & & \\ & I_{j-i} & & & & \\ & & 1 & 0 & 0 & \\ & & t & -t & 1 & \\ & & 0 & 0 & 1 & \\ & & & & & I_{n-j-2} \end{bmatrix}.$$

If we let

$$A = \begin{bmatrix} 1 & 0 & 0 & \\ t & -t & 1 & \\ 0 & 0 & 1 & \\ & & & I_{n-i-2} \end{bmatrix}$$

and

$$B = \begin{bmatrix} I_{j-i} & & & & \\ & 1 & 0 & 0 & \\ & t & -t & 1 & \\ & 0 & 0 & 1 & \\ & & & & I_{n-j-2} \end{bmatrix},$$

then the problem becomes equivalent to showing that $AB = BA$. If $j - i \geq 3$, then we can re-write B as

$$B = \begin{bmatrix} 1 & 0 & 0 & & & & & \\ 0 & 1 & 0 & & & & & \\ 0 & 0 & 1 & & & & & \\ & & & I_{j-i-3} & & & & \\ & & & & 1 & 0 & 0 & \\ & & & & t & -t & 1 & \\ & & & & 0 & 0 & 1 & \\ & & & & & & & I_{n-j-2} \end{bmatrix},$$

and from here we obtain that

$$AB = \begin{bmatrix} 1 & 0 & 0 & & & & & \\ t & -t & 1 & & & & & \\ 0 & 0 & 1 & & & & & \\ & & & I_{j-i-3} & & & & \\ & & & & 1 & 0 & 0 & \\ & & & & t & -t & 1 & \\ & & & & 0 & 0 & 1 & \\ & & & & & & & I_{n-j-2} \end{bmatrix} = BA.$$

If $j - i < 3$, then $j - i = 2$ and $j = i + 2$. This gives us

$$B = \begin{bmatrix} 1 & 0 & & & & \\ 0 & 1 & & & & \\ & & 1 & 0 & 0 & \\ & & t & -t & 1 & \\ & & 0 & 0 & 1 & \\ & & & & & I_{n-j-2} \end{bmatrix}$$

and we re-write A as

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ t & -t & 1 & 0 \\ & & I_{n-i-1} & \end{bmatrix},$$

where note that the entry 1 above I_{n-i-1} is singular. We thus conclude that

$$AB = \begin{bmatrix} 1 & 0 & 0 & & & & \\ t & -t & 1 & & & & \\ & & 1 & 0 & 0 & & \\ & & t & -t & 1 & & \\ & & 0 & 0 & 1 & & \\ & & & & & & I_{n-j-2} \end{bmatrix} = BA,$$

and our proof of the first property is complete.

For the proof of the second property, we distinguish the following three cases based on whether $i = 1, i = n-2$ or $2 \leq i \leq n-3$.

1. If $i = 1$, then we have

$$\psi_r \sigma_1 = \begin{bmatrix} -t & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{bmatrix}$$

and

$$\psi_r\sigma_2 = \begin{bmatrix} 1 & 0 & 0 & \\ t & -t & 1 & \\ 0 & 0 & 1 & \\ & & & I_{n-4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \\ t & -t & 1 \\ & & I_{n-3} \end{bmatrix},$$

where the entry 1 right above the block matrix I_{n-3} within $\psi_r\sigma_2$ is singular. By applying block matrix multiplication, we obtain

$$\psi_r(\sigma_1\sigma_2\sigma_1) = (\psi_r(\sigma_1\sigma_2))(\psi_r(\sigma_1)) = \begin{bmatrix} 0 & -t & 1 & 0 \\ t & -t & 1 & 0 \\ & & I_{n-3} & \end{bmatrix} \begin{bmatrix} -t & 1 & \\ 0 & 1 & \\ & & I_{n-3} \end{bmatrix} = \\ = \begin{bmatrix} 0 & -t & 1 & 0 \\ -t^2 & 0 & 1 & 0 \\ & & I_{n-3} & \end{bmatrix}$$

,

where the two 1 entries above the $I_{n-3}-$block are both singular. Again by applying block matrix multiplication, we get

$$\psi_r(\sigma_2\sigma_1\sigma_2) = (\psi_r(\sigma_2))(\psi_r(\sigma_1\sigma_2)) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ t & -t & 1 & 0 \\ & & I_{n-3} & \end{bmatrix} \begin{bmatrix} 0 & -t & 1 & 0 \\ t & -t & 1 & 0 \\ & & I_{n-3} & \end{bmatrix} = \\ = \begin{bmatrix} 0 & -t & 1 & 0 \\ -t^2 & 0 & 1 & 0 \\ & & I_{n-3} & \end{bmatrix},$$

where the entry 1 above the $I_{n-3}-$block in the first matrix is singular, and the two entries 1 above the $I_{n-3}-$block in each of the following matrices are both singular as well. We conclude that $\psi_r(\sigma_1\sigma_2\sigma_1) = \psi_r(\sigma_2\sigma_1\sigma_2)$ and so $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.

2. If $i = n-2$, then

$$\psi_r\sigma_{n-2} = \begin{bmatrix} I_{n-4} & & & \\ & 1 & 0 & 0 \\ & t & -t & 1 \\ & 0 & 0 & 1 \end{bmatrix}$$

and

$$\psi_r\sigma_{n-1} = \begin{bmatrix} I_{n-3} & & \\ & 1 & 0 \\ & t & -t \end{bmatrix} = \begin{bmatrix} I_{n-4} & & & \\ & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & t & -t \end{bmatrix}.$$

Block matrix multiplication gives us

$$\psi_r(\sigma_{n-2}\sigma_{n-1}\sigma_{n-2}) = (\psi_r(\sigma_{n-2}\sigma_{n-1}))(\psi_r(\sigma_{n-2})) = \begin{bmatrix} I_{n-4} & & & \\ & 1 & 0 & 0 \\ & t & 0 & -t \\ & 0 & t & -t \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} I_{n-4} & & & \\ & 1 & 0 & 0 \\ & t & -t & 1 \\ & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I_{n-4} & & & \\ & 1 & 0 & 0 \\ & t & 0 & -t \\ & t^2 & -t^2 & 0 \end{bmatrix}.$$

and

$$\psi_r(\sigma_{n-1}\sigma_{n-2}\sigma_{n-1}) = (\psi_r(\sigma_{n-1}\sigma_{n-2}))(\psi_r(\sigma_{n-1})) = \begin{bmatrix} I_{n-4} & & & \\ & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & t & -t \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} I_{n-4} & & & \\ & 1 & 0 & 0 \\ & t & 0 & -t \\ & 0 & t & -t \end{bmatrix} = \begin{bmatrix} I_{n-4} & & & \\ & 1 & 0 & 0 \\ & t & 0 & -t \\ & t^2 & -t^2 & 0 \end{bmatrix},$$

which proves that $\psi_r(\sigma_{n-2}\sigma_{n-1}\sigma_{n-2}) = \psi_r(\sigma_{n-1}\sigma_{n-2}\sigma_{n-1})$ and so

$$\sigma_{n-2}\sigma_{n-1}\sigma_{n-2} = \sigma_{n-1}\sigma_{n-2}\sigma_{n-1}.$$

3. If $2 \leq i \leq n-3$, then we have

$$\psi_r\sigma_i = \begin{bmatrix} I_{i-2} & & & & \\ & 1 & 0 & 0 & \\ & t & -t & 1 & \\ & 0 & 0 & 1 & \\ & & & & I_{n-i-2} \end{bmatrix}$$

and

$$\psi_r\sigma_{i+1} = \begin{bmatrix} I_{i-1} & & & & \\ & 1 & 0 & 0 & \\ & t & -t & 1 & \\ & 0 & 0 & 1 & \\ & & & & I_{n-i-3} \end{bmatrix}.$$

This time, we break our matrices into three blocks, and write

$$\psi_r \sigma_i = \begin{bmatrix} I_{i-1} & & & & \\ & t & -t & 1 & \\ & & 0 & 1 & \\ & & & & I_{n-i-2} \end{bmatrix},$$

where the entry t below the block I_{i-1} is singular, and

$$\psi_r \sigma_{i+1} = \begin{bmatrix} I_{i-1} & & & & \\ & 1 & 0 & & \\ & t & -t & 1 & \\ & & & & I_{n-i-2} \end{bmatrix},$$

where the entry 1 above the block matrix I_{n-i-2} is also singular. By writing the matrices in this way, block matrix multiplication gives us

$$\psi_r(\sigma_i\sigma_{i+1}\sigma_i) = (\psi_r(\sigma_i\sigma_{i+1}))(\psi_r(\sigma_i)) = \begin{bmatrix} I_{i-1} & & & & \\ t & 0 & -t & 1 & \\ & t & -t & 1 & \\ & & & & I_{n-i-2} \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} I_{i-1} & & & & \\ t & -t & 1 & & \\ & 0 & 1 & & \\ & & & & I_{n-i-2} \end{bmatrix} = \begin{bmatrix} I_{i-1} & & & & \\ t & 0 & -t & 1 & \\ t^2 & -t^2 & 0 & 1 & \\ & & & & I_{n-i-2} \end{bmatrix},$$

where in the first matrix, the entry of t below the block matrix I_{i-1} and the two entries 1 above the block matrix I_{n-i-2} are singular, in the second matrix, the entry of t below I_{i-1} is singular, and finally, in the third matrix, the two entries t and t^2 below the block matrix I_{i-1} and the two entries of 1 above the block matrix I_{n-i-2} are singular. At the same time, we can write

$$\psi_r(\sigma_{i+1}\sigma_i\sigma_{i+1}) = (\psi_r(\sigma_{i+1}\sigma_i))(\psi_r(\sigma_{i+1})) = \begin{bmatrix} I_{i-1} & & & & \\ & 1 & 0 & & \\ & t & -t & 1 & \\ & & & & I_{n-i-2} \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} I_{i-1} & & & & \\ t & 0 & -t & 1 & \\ & t & -t & 1 & \\ & & & & I_{n-i-2} \end{bmatrix} = \begin{bmatrix} I_{i-1} & & & & \\ t & 0 & -t & 1 & \\ t^2 & -t^2 & 0 & 1 & \\ & & & & I_{n-i-2} \end{bmatrix},$$

where in the first matrix, the entry of 1 above the block matrix I_{n-i-2} is singular, in the second matrix, the entry of t below the block matrix I_{i-1} and the two entries of 1 above the block matrix I_{n-i-2} are singular, and finally, in the third matrix, the entries of t and t^2 below the block matrix I_{i-1} and the two entries of 1 above the block matrix I_{n-i-2} are also singular. This shows that

$$\psi_r(\sigma_i\sigma_{i+1}\sigma_i) = \psi_r(\sigma_{i+1}\sigma_i\sigma_{i+1}),$$

or $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, which completes our proof.

□

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