Víctor Neumann-Lara
The Dichromatic Number of a Digraph

A Mathematics Thesis

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Introduction and Purpose

Mathematics has been perceived through an absolutist lens by the majority of people today, including mathematicians. In the philosophy of mathematics, the absolutist views mathematics as an "objective, absolute, and incorrigible body of knowledge, which rests on the firm foundations of deductive logic". This causes mathematics to be"timeless, ahistorical, ... and both free of value and culture"[1]. A reason why one could fall into this way of thinking is due to how mathematics can reflect the universe and occur outside human influences.[2] Although there are mathematical truths that are indisputable, the mathematics community should not accept that mathematics is universal and culture-free. Recent research in mathematical education, starting in the 1970's, have shown the growing need to part from this perspective[3] as mathematics holds a history of elitism and hegemony that often goes undiscussed, unnoticed, and unidentified.[4] In fact, it can be argued that modern [western] mathematics is Eurocentric, Americentric, and exclusionary.[5] From those who accept the view that mathematics is objective, they are made blind to other perspectives from the variety of cultures that have contributed to mathematics in unique and important ways.

This thesis attempts to break through Eurocentric mathematics and focuses on a non-European mathematician through an *ethnomathematics* approach. Ethnomathematics falls under ethnoscience, which D'Ambrioso defines as the study of the scientific and technological phenomena through its social, economic, and cultural background.[6] This term was introduced in the late 1970's [7] and has since grown into a unique field of study for the sciences, mathematics, mathematical history, and cultural anthropology. In this paper, the mathematician chosen is one who is not widely discussed in the math community. In this paper, we explore this mathematician's selected developments in advancing mathematics.

I will draw on original sources and trace the relationships among different fields within mathematics through an in-depth study. The mathematician selected is Víctor Neumann-Lara and his contributions to graph theory. In this thesis, I will provide a biography which discusses the contributing factors of Neumann's growth. I will briefly discuss a history of the Instituto de Matématicas at UNAM as the institution provides valuable information for some of Víctor's influences. Then, I will dive into his paper, "The Dichromatic number of a Digraph" (1982), that influenced much of his further work. Neumann-Lara's impact on the mathematical community will be discussed.

[1] Refer to [24]

[2] Refer to pp.18 of [23]

[3] Refer to [7],[17],[29], and [45]

[4] Wynter argues the Euroamerican dominant culture has influenced and even created modern science practices to incorporate racism and discrimination. Refer to pp.43-44 of [62]

[5] Refer to pp.51 of [7]

[6] Refer to pp.44 of [17]

[7] Refer to [53]

Biography, 1933-1982

Víctor Manuel Neumann-Lara was born on June 6, 1933, the youngest child to his parents Max Neumann Hufnagel and Carolina Lara . He was born in Huejutla, Hidalgo, México, a place of a variety of cultures, including that of the Indigenous people where the native language Nahuatl is spoken (a language Neumann also spoke). Max Neumann was born in Sztetin, Prussia (now Poland), but grew up in Germany. His father first made his way to México when he was 18 in 1921, following the end of World War I when conditions in Germany were horrible. Max arrived in Tampico, México and was given a few days off the ship before it would continue its journey back. However, Max never returned to the ship and he never returned back to Germany. Max found a living operating the first electric power generator in the region. Carolina Lara and her family were from Huejutla, Hidalgo, as they had descended from generations staying in the region. In fact, Carolina's paternal grandmother was a Huasteca Indian. Max and Carolina together had four children, Francisco, Martha, Raúl, and Víctor.

In 1942, Víctor's mother decided to move his siblings and him to México City for a couple of reasons. One reason was due to the lack of education past fifth grade in their area. The other reason was because his father had become paralyzed in his legs and couldn't find work, thus making his mother the breadwinner of the household. When the family decided to move, Max chose to remain in Huejutla. Once in México City, Víctor recieved a scholarship from the Secretary of Public Education (SEP) from elementary school to high school, where Víctor attended the Escuela Secundaria[8] No. 4. His literature teacher in high school was Carlos Pellicer Cámara[9], a very well known Mexican poet, who would later become a Mexican senator. Pellicer greatly influenced Víctor and helped to develop his interest in poetry. After completing high school, Víctor worked for four years for the SEP. Víctor was miserable in his job at SEP doing bureaucratic work and his intermediate boss, engineer Paliza, obtained a permit for Víctor to practice math problems with him to ease Víctor's work load.

Paliza helped to kindle Víctor's passion for mathematics, although Víctor did not know mathematics would become his career yet. After finishing working at SEP for four years, Víctor enrolled at the Instituto Politénico Nacional (IPN)[10] in petroleum engineering for two years in hopes of studying more mathematics. However, there was little if any math involved with those studies; they focused more on topography, which Víctor did not like. He then enrolled into the Escuela Superior de Ingeniería Mecánica y Eléctrica (ESIME)[11] in hopes of studying more mathematics, but to no avail. Víctor then decided to study math on his own. Around this time, Víctor ran into a former

[8] High School No. 4. [47]

[9] According to one source Pellicer taught at the high school level for 20 years, but there are not many sources to back that claim up. [12]

[10] The National Polytechnic Institute located in México city.

[11] The Higher School of Mechanical and Electrical Engineering located at IPN.

teacher from IPN at a bookstore and asked him where he could study mathematics. The former teacher advised Víctor to see Vicente Echeverría del Prado, another teacher at IPN who taught mathematics and was also an architect and poet. Vicente then introduced Víctor to Francisco Zubieta, a distinguished professor in the Faculty of Sciences at the Universidad Nacional Autónoma de México (UNAM)[12] who taught courses at IPN. They met up in the Mining Palace on the main campus of UNAM, where the Faculty of Sciences was located. By 1953, Víctor got to meet Alfonso Nápoles Gándara[13], Félix Recillas Juárez[14], and other pioneering Mexican mathematicians. (There is more information concerning these mathematicians below in the *History of UNAM* section.)

Víctor soon enrolled in UNAM in the Faculty of Sciences in 1954 with a class of 25 students, who held a variety of interests relating to mathematics such as physics and actuarial sciences. The way Víctor learned about the Faculty of Science through networking was typical as it was not well known. In fact, many people had no idea that mathematics was a serious subject of study in México and would confuse Víctor's study with accounting. At UNAM, Víctor studied under Alberto Barajas (PhD in mathematical sciences from Harvard University), Guillermo Torres (who later became Víctor's thesis advisor), and Gonzalo Zubieta to name a few. His passion for mathematics developed greatly, especially in the fields of geometry and topology. Barajas soon recommended that Víctor become an assistant professor to Javier Barros Sierra in teaching Algebra. When Barros became director of the Faculty of Engineering in 1956, Barros offered Víctor a job as a teacher at the university. Víctor accepted and worked there while continuing his undergraduate studies in mathematics, which he completed in 1958.[15] It would be until 1966 when Víctor published his undergraduate thesis titled, "Equivalencia Entre La Homologia Cubica y la Simplicial"[16].

In 1959, the mathematician Federico Velasco[17] recommended Víctor for a position at the Universidad Central de Venezuela[18]. Víctor accepted the position at the university to become a full-time professor and remained there for three years. While working there, he met Carmen Coto and married her in 1960. They had three children together: Max (1962)(named after Víctor's father), Guillermo (1963), and Citlalin (1974). Al-

[12]The National Autonomous University of Mexico located in Mexico City with satellite campuses located throughout the world.

[13]Refer to History of UNAM section on page 6.

[14]Professor at the Institute of Mathematics at UNAM in Algebraic Topology with degrees from Princeton University who also advocated for Mexican mathematics. [3]

[15]Neumann did not complete a PhD program because at that time in México it was not required to obtain a doctorate to teach at a university. The only thing required to teach was a recommendation by a professor. [37][52][47]

[16]Refer to Appendix [2]

[17]Originally, Federico Velasco was from Spain and escaped to México from the White Terror under dictator Francisco Franco during the Spanish Civil War. The Mexican President opened México's doors to the fleeing Spanish and encouraged them to contribute back to Mexican society. He later traveled with Víctor to Veracruz and back to México after the events in Venezuela in 1962. [52]

[18]Central University of Venezuela located in the city of Caracas.

though Víctor enjoyed his time in Venezuela, the Cold War raging between the United States and the U.S.S.R. in Central and South America would interrupt his stay. Tensions were building among the superpowers and by the time the Cuban Missile crisis was full blown in October 1962, the Venezuelan president, Rómulo Betancourt (a leftist but anti-communist leader), had responded by repressing the Movimiento de Izquierda Revolucionaria (MIR)[19] and other leftist movements that had ties with the communist party. Víctor held leftist views but didn't participate in any communist activities in Venezuela. However, Carmen's family was heavily involved in politics, particularly with their affiliation with the communist party. Víctor was jailed for 12 days in September of 1962, two months after Max was born, for his connection to the Communist party through his wife's family. Víctor promptly left back to México with his new family after being released from jail that same year.

Víctor accepted a teaching position upon returning to México in 1962 as a professor at the Universidad Veracruzana[20] in the school of Physics and Mathematics. He was promoted to director of the department that same year and remained at the university until 1966. During his time there, Víctor was able to study in Paris at the Université de Paris and Université Clermont Auvergne[21] in 1964. In Paris, he studied alongside a variety of mathematicians in algebra and combinatoric logic. The most notable were Roland Fraïssé[22] and Mark Krasner[23]. During this time, Víctor also obtained his masters at the Université Clermont Auvergne in 1965.[24]

It is hard to say how Víctor first learned about graph theory, but the first exposure could have been in his undergraduate studies under his thesis advisor's, Guillermo Torres. Torres worked in his interests of geometry, topology, knot theory, and combinatorics. Víctor returned from Paris in 1966 and brought back a new variety of interests in graph theory: clique graphs, kernels of digraphs, dichromatic number, and acyclic sub-tournaments. México did not have many of its own mathematicians doing research at this time in the 1960's, let alone research outside of algebra and topology. To make up for this, Víctor studied graph theory on his own using Claude Berge's[25] books in graph theory. (Berge and Víctor would have an interaction which is mentioned later in this paper). Víctor made such great progress in studies that he was able to bring on a

[19] The Revolutionary Left Movement (in English) holding more radical left beliefs than the Acción Democrática (Democratic Action) party and actually split from AD in 1959. [50]

[20] Veracruzana University located in Xalapa, Veracruz, México.

[21] The University of Paris and the University of Clermont-Ferrand. [37]

[22] Professor in Mathematical Logic from L'Université d'Aix-Marseille (University of Provence) located in Aix-en-Provence and Marseille, France. [11]

[23] Professor in algebraic number theory from the University "Pierre et Marie Curie" (University Pierre and Marie Curie, Paris University Campus VI) located in Paris, France. [58]

[24] Refer to [37]

[25] Professor in combinatorics from the University of Paris and known as a founder of modern combinatorics and graph theory. [14][44]

student, Fernando Escalante[26], later in 1969.

In 1966, Víctor accepted a new position at UNAM as a full-time professor in the Faculty of Sciences. Although Víctor was no longer the director at the Universidad Veracruzana, he still taught two courses there for the next two years. He remained in the Faculty of Sciences at UNAM until April 1, 1980 when he moved to the Instituto de Matemáticas[27]. Víctor would remain at the institute for the rest of his life, where the majority of his work would be done.

The first contribution Víctor made to a publication was from his paper, "Seminucleus of a Digraph", published in 1971 under the Institute of Mathematics at UNAM. However, what pushed Víctor to continue publishing his work was from his contribution to the book "Graphs and Hypergraphs"[28] by Claude Berge in 1973. In Víctor's contribution, he simplified Moses Richardson's theorem that proved a graph without odd circuits possesses a kernel. Although a tiny contribution in the book, the expressed interest by the author to include his work motivated Víctor to publish more of his own work and pursue more publications. He started publishing regularly with his paper "Mengerian theorems for paths of bounded length" in 1978 in collaboration with other mathematicians, such as his students and colleagues. By 1982, Víctor published "The Dichromatic Number of a Digraph"[29].

UNAM and Instituto de Matemátics

A Brief History leading to Víctor Neumann-Lara

UNAM was first founded on September 21, 1551 under the original name of the Royal and Pontificial University of México. By the 1700's, there were many new schools that had been founded and started to expand. In 1910, all the universities were placed under Mexican government control as the Mexican Revolution began that same year and integrated them into one school called the Universidad Nacional de México. The original campus was then destroyed that same year. The newly integrated school was a complete change from what it orginially was under Mexican leadership. In 1929, the school was given autonomy and became what is known today as the Universidad Nacional Autónoma de México. In 1954, UNAM's main campus was relocated and rebuilt, and named "Ciudad Universitaria", translated as University City, and continues to be called that today.

Mathematics has been taught at UNAM for a long time. In the early 1900's, it was taught in the schools of Higher Studies and Engineering. The School of High Studies became the Faculty of Philosophy & Letters in 1924 with the leadership of philosophy

[26]No information could be found on Fernando Escalante
[27]Mathematics Institute located on the main campus of UNAM
[28]Refer to [5].
[29]Refer to [29]

professor Antonio Caso, which did not alter the studies in the sciences or math. The autonomy given to the university in 1929 then allowed for the creation of the science section of the faculty to emphasize more on those studies. Mathematics continued to grow in the 1930's and by 1934, Manuel Gómez Morín[30] and Ricardo Monges López proposed to the university a new school, the School of Physical & Mathematical Sciences. The proposed school began to operate on March 1^{st}, 1936 from the efforts of Monges to promote the school. At the end of the following year, in December 1937, a new proposal was made to the university for an Institute of Physical & Mathematical Research, which started work soon after in February 1938 under the direction of Monges[31]. The research institute was approved January 1939 by UNAM, which was the same month that also saw the initiation of activities for the new Faculty of Science.

The Institute of Physical & Mathematical sciences officially began operations June 30, 1942 under the guidance of Dr. Alfonso Nápoles Gándara[32], which oversaw the formal beginning of the Institute of Mathematics. The Institute was first housed in the Palacio de Minería (Mining Palace) in México City where the National School of Engineers, Faculty of Sciences, and Institute of Physics were also placed. The institute was divided into three departments: Applied Mathematics headed by Roberto Vásquez, Logic & Fundamentals head by Francisco Zubieta, and Pure Mathematics head by Alberto Barajas. In 1953, the institute was moved to the main campus, Ciudad Universidad, where it is found today.

An important figure to mention is the self taught mathematician[33], Soterto Prieto Rodríguez (1884-1935). He was a highly respected teacher and mathematician, specializing in geometry and algebra, who influenced many future Mexican mathematicians. In 1910, Prieto taught at the the National Preparatory School, the National School of Engineers, and the National School of Higher Studies. Once the National University of Mexico became established, Prieto, who was only 27 years old, was appointed to teach the first course in higher mathematics in México in the "Theory of Analytical Functions". Prieto recognized the importance of furthering mathematics in México through its study and advanced research. He began teaching mathematics at UNAM in 1932 and simultaneously that same year created the "Antonio Alzate" group as a part of the Mathematics Section of the Scientific Society from the Academy of sciences as a space for fellow Mexican mathematicians to convene and discuss mathematics. This group became the foundation for the Sociedad de Matemáticas Mexicana (SMM)[34]. SMM is an organization where many prominent Méxican mathematicians were members and continues to this day. One such mathematician who was a part of SSM and an influence for Víctor was Alberto Barajas.

[30] Student in law and Rector of the university who would later become a lawyer and politician. [31]
[31] Refer to [1], [31] & [41].
[32] MIT and UNAM graduate in mathematics. [31]
[33] Refer to [41]
[34] Translates to the "Society of Mexican Mathematicians". [55]

Alberto Barajas Celis entered the National Prepatory School in 1930 and studied under Sotero Prieto and Alfonoso Nápoles in the engineering track by 1932.[35] However, when Barajas was taking classes, advanced math courses were seldomly offered. Prieto would offer these courses without credit, which Barajas and his peers (Carlos Graef Fernández and Nabor Carrillo) would take without hesitation. Barajas and Graief decided to dedicate their professions to mathematics in 1934 and Barajas became a teacher at the National Prepatory School that same year. By 1938, Barajas became a professor in the Faculty of Sciences[36]. He would then hold multiple teaching and supervising positions in the Institute of Mathematics and Faculty of Sciences between 1947 and 1969. One of Barajas students was Víctor Neumann-Lara when Víctor took his courses in the 1950's.

In the biography of Víctor, we know that he studied under Barajas and learned geometry from him[37] and Barajas grew to admire Víctor as his student and recommended that he become an assistant professor. The two became life long friends. Víctor admired Barajas so much that he dedicated his paper, "*The Dichromatic Number of a Digraph*", to Barajas!

Influences on the *Dichromatic Number of a Digraph* paper

Paul Erdös enters the Picture

Víctor Neumann-Lara's work on the dichromatic number of a digraph first began with his interest in graph theory. He studied graph theory on his own time and learned a lot from Berge's graph theory textbook, but that wasn't the sole book he used to teach himself. Frank Harary[38] published a textbook in 1969 titled,"Graph Theory"[39]. Neumann utilized this book for his paper and it is likely this book helped to form Neumann's idea for the dichromatic number. Harary discusses a range of topics in graph theory, which Víctor probably focused on the chapters *Planarity, Colorability*, and *Digraphs*. Harary also highlights the influences Erdös had in graph theory. One such influence is Vollmerhaus's work on an Erdös conjecture proving there is a finite collection of forbidden subgraphs (subgraphs not allowed in the collection) on any embedded orientable surface. Harary mentions Erdös multiple times in various chapters, which could be an additional reason for Neumann's admiration of the Hungarian mathematician. Beyond Harary's textbook, Erdös worked on chromatic theory of graphs. Neumann references to a paper Erdös wrote in 1966 with András Hajnal titled, *On Chromatic Number of Graphs and Set-Systems.*[40]

[35]Refer to [31],[32],[41].
[36]Refer to [40].
[37]Refer to [47].
[38]Ph.D. in Mathematical Logic from UC Berkeley. Considered one of fathers of Modern Graph theory he helped found and revitalize. Refer to [43].
[39][30]. Considered modern classic that helped establish terminology for graph theory. [43]
[40]Refer to [22].

Neumann developed more work on the dichromatic number when he first invited Paul Erdös in 1977 to UNAM. From 1977 to 1979, both Neumann and Erdös worked on the dichromatic number of a digraph. In 1979, Erdös published some of the problems he was working on at a numerical conference[41], which included the work he and Neumann had done. According to Erdös's paper, Neumann seemed to be aware of the problem before inviting Erdös to México with his original definition of the dichromatic number.

> "The dichromatic number of G [is] the smallest integer so that the vertex set of a digraph can be decomposed into $d_k(G)$ [the dichromatic number of a digraph] disjoint subsets none of which span a directed circuit"[42].

This original definition is what Erdös and Neumann used to start their investigation into modifying the function of $d_k(G)$.

The first modification they made is to define the dichromatic number for a *graph* instead of a *digraph*; they do this by redefining G as a graph, rather than a digraph that they had established earlier in the paper. The next and final modification they made was to add the importance of the orientation of the edges of G. The resulting modified definition Erdös and Neumann end up with is the following

> Let G be (an undirected) graph. The dichromatic number $d_k(G)$ is the smallest integer so that for any orientation of the edges of G one can always divide the vertex set into $d_k(G)$ or fewer disjoint sets, none of which span a directed circuit of G (in the given orientation).

This addition of orientation is important because this creates a lot of issues for both Erdös and Neumann as they cannot determine $d_k(G)$ for simple graphs. The closest thing they were able to determine was for complete graphs $k(n)$ by proving the following.[43]

$$\frac{c_1 n}{log\ n} < d_k(k(n)) < \frac{c_2 n}{log\ n}$$

This equation shows that there is a bound that was constructed to describe the dichromatic number. The upper and lower bound for the dichromatic number is $\frac{n}{logn}$ with the coefficients of c_1 and c_2. It is not clear what c_1 and c_2 represent, but it is possible they represents some particular orientations that lead to substantial values that create the bounds.

Neumann later published his paper of his individual findings in 1982 on the dichromatic number of a digraph. The terminology that remained the same was for the

[41] Refer to pp.17 in Erdös paper. [21]

[42] Specific terminology and definitions will be discussed in the *Dichromatic Number of a Digraph* section

[43] The k in $d_k(G)$ and $k(n)$ are not the same.

dichromatic number, d_k. However, Neumann changed some of the terms (probably to avoid confusion). Originally, Neumann and Erdös denoted a graph as $G(n; e)$ with n representing vertices and e for edges. $G(n)$ represented a graph of n vertices and G_e represented a graph with e edges. Digraphs were denoted as G. He changes the terms so $G(n; e)$ was not utilized, G means an undirected graph and D for digraphs, $V(G)$ or $V(D)$ for vertices of a graph or digraph, and similarly for arcs $A(G\ or\ D)$ and edges $E(G\ or\ D)$. The problem that Erdös and Neumann ran into with to prove the dichromatic number of a digraph with orientation was something Neumann managed to solve in theorems 4-6.

Unfortunately, Neumann was unable to co-author a paper with Erdös on the topic, for reasons unknown to me. Specifically, on the concept of the dichromatic number where

$$d_k(G) = max\{d_k(G_*)|G_* \text{ is an orientation of G}\}.$$ [44,45]

In simple terms, the dichromatic number of a graph (not a digraph) is equal to the amount of different dichromatic numbers taken from the different orientations of a given graph. Since Neumann did not publish a formal paper with their findings, Neumann receives an Erdös number of two[46]. Although, I would argue that Neumann should have an Erdös number of 1 due to Erdös publishing a paper that includes Neumann's work.

Before this section finishes, a brief remark should be made on Erdös' visit to México. In an article written by Dr. Alfinio Flores, Erdös's first trip to México in 1977 involved a hike at a nearby mountain.[47] Flores and five other mathematics students joined Erdös on the hike, unfortunately Neumann was unable to join due to health issues. The hike went very well and Erdös, in his mid-sixties, kept a steady pace and only needed help going down the mountain. The whole way up Erdös talked and did not stop for a break until he reached the summit. According to Dr. Isabel Puga, Erdös would continue to visit México a few times in the late 1980's to early 1990's to work with the Combinatorics group that Neumann had founded. Puga and other mathematicians took Erdös to visit the archaeological site of Teotihuacán. However, Erdös was lost in his thoughts of math and paid little attention to Teotihuacán.[48]

[44] Neumann also briefly covers a similar topic in his paper with the first corollary he writes in his paper. Refer to pp.266, 270 of [34].

[45] This concept was also briefly covered in Erdös's paper. Refer to pp.21 of [19]

[46] Refer to [20].

[47] According to Flores, Erdös enjoyed hiking at nearby mountains on his travels. [25]

[48] Refer to [48].

Dichromatic Number of a Digraph

In this paper, only the first definition will be discussed from Neumann's paper as the rest of his paper discusses additional results he made from the dichromatic number and chromatic theory. However, before explaining the dichromatic number of a digraph, the foundations of graph theory will be discussed. Terms that are bold represent the terms pertinent to the paper.

Basics of Graph Theory

Graphs

A **graph** G consists of two finite nonempty sets: V, E. This can be written as $G = (V, E)$. The set $V(G)$ consists of elements called **vertices** V, which are the points of a graph. The set $E(G)$ consists of elements called **edges** E, the unordered pairs of elements from $V(G)$. An edge can also be presented as $(u, v) \in E$. A vertex is also known as a node, but the term vertex will be used for the purposes of this paper. A graph contains a *loop* when an edge that joins a single endpoint to itself. In this paper, loops will not be considered for graphs or digraphs unless otherwise stated. When a graph $G = (V, E)$ contains some deletions in V and E, it creates a **subgraph** $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$.

A *path* is a G containing v vertices and $v - 1$ edges, such that no vertices are repeated in G. The first and last vertices are called the endpoints of the path. A graph is *connected* if a path can be created between any vertices in G such that the vertices are endpoints; this path is a subgraph of G. A **cycle** is created when the endpoints of a path are connected. A graph is considered **acyclic** when a cycle cannot be constructed in G.

Graph Coloring

The coloring of a graph is the assignment of colors to each vertex or edge of an undirected graph G and is considered to be **k-colorable** when there exists at most k colors. Colors can be presented as actual colors, but are typically shown as numbers that each represent a different color. As there are two ways to color a graph, this paper will concentrate on the *vertex coloring* of a graph and will refer to it solely as the coloring of a graph throughout this paper unless otherwise noted. The colors that are assigned to the vertices are taken from a set C of colors: $C = \{1, 2, 3, ..., k\}$ when the graph is colored with k colors. A subset of vertices describing an independent group of colors, in a proper colored graph, is called a chromatic class[49]. Chromatic classes can be represented as $V_1, V_2, ..., V_k$. A *proper coloring* is where each vertex will be assigned a color so no adjacent vertex (connected by an edge) is the same color. The **chromatic**

[49] A *class* of graphs is a group of graphs that share a common attribute; e.g. bipartite graphs, tree graphs, digraphs, et cetera.

number of a graph $\chi(G)$ is the least number of k colors in the proper coloring of a graph. This means a graph G with $\chi(G)$ will also be a k-chromatic graph or simply *k-chromatic*. An example is provided with *Figure 1*.

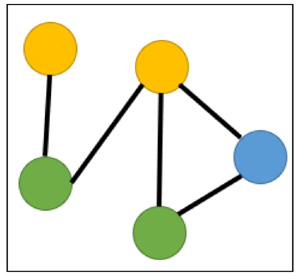


Figure 1
Graph G

In *Figure 1*, G is 3-colorable and will also have $\chi(G) = 3$. This means that since G is a proper coloring, then G is k-chromatic.

Digraphs

A directed graph D is also called a **digraph**, which is the term we will use for the remainder of this paper. Simply put, a digraph is when a graph's edges are assigned directions and is denoted as $D = (V, A)$**.** D is comprised of two finite nonempty sets: V, A. Much like in a graph, the set $V(D)$ consists of vertices. The set $A(D)$ is comprised of the ordered pairs of elements from $V(D)$ and are called **arcs**, which are also directed edges. Arcs can be denoted in two ways. (i) The arc $(u, v) \in A(D)$ assigns the direction of the arc from vertex u to vertex v. (ii) The arc $uv \in A(D)$ assigns the direction of the arc from vertex u to vertex v. An arc is represented in graph as an edge with an arrow pointing in the direction it has been assigned. A digraph is considered a *complete digraph* when $A(D)$ contains a pair of symmetric arcs uv and vu between any pair of vertices u, v from $V(D)$. A digraph that corresponds to a graph is notated as $D(G)$.

A *walk* in a digraph is D where an arc connects any two consecutive arcs to create a finite sequence of arcs, where vertices may be repeatedly used. A *directed path* contains v vertices and $v - 1$ arcs, such that no vertices are repeated in D and creates two endpoints. A digraph is *connected* if an undirected path can be created between any vertices in D. A digraph is *strongly connected* if a directed path can be created between any vertices in D. A *directed cycle* exists when a directed path is connected by their endpoints. An *acyclic digraph* exists when no directed cycles exist in D. A **subdigraph** is similar to the subgraph of G. When a digraph $D = (V, A)$ deletes some V and A, a subdigraph $D' = (V', A')$ such that $V' \subseteq V$ and $A' \subseteq A$.

An orientation of a (undirected) graph is the assignment of a direction to each edge $(u, v) \in G$, effectively replacing each edge with an arc uv or $vu \in A(G)$. An oriented graph is denoted as $O(G)$ and is a type of digraph where there are no symmetric pair of arcs (e.g. the arcs u, v and vu cannot both exist between $(u, v) \in A(D)$. This paper will refer to oriented graphs as digraphs unless otherwise specified.

Coloring Oriented Graphs

Applying the chromatic number to a digraph first starts with applying it to oriented graphs. Sopena[50] gives a nice definition for oriented colorings. An *oriented k-coloring* of an oriented graph is a partition of vertices in $V(G)$ into at most k color classes such that it satisfies the following two properties.

(i) Any pair of adjacent vertices don't belong in the same color class V_n. $u, v \in V(G)$. $(u, v) \in E(G)$. $u \in k_1$ and $v \in k_2$ where $u, v \notin V_n$ (ii) All arcs linking two color classes have the same direction.
$u, v, x, y \in V(G)$. $uv \in A(G)$. $xy \in A(G)$. $u, y \notin C_1$. $v, x \in C_2$.

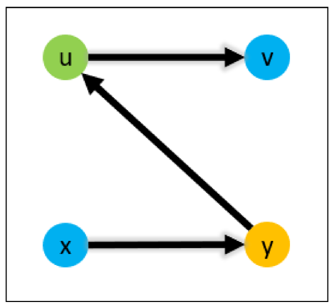


Figure 2
Property (ii) of an oriented k-coloring

The *oriented chromatic number* $O\chi(G)$ is the smallest k such that $O(G)$ will possess an oriented k-coloring. $O\chi(G)$ also applies to undirected graphs G by being the largest $O\chi(G)$ of the different orientations comprised from G.

Digraph Coloring

Colors are assigned to the vertices of a digraph. A digraph is k-colorable when there exists at most k-colors. Colors assigned to the vertices are taken from a set I_n of colors: $I_n = \{1, 2, 3, ..., n\}$. (There is a shift of notation from using k to n to represent colors.) Color classes of a digraph are the set of vertices with the same color, $I_{v1}, I_{v2}, ..., I_{vn}$.

A proper n-coloring of a digraph is (i) when any two adjacent vertices connected by a pair of symmetric arc are different colors and (ii) the vertices in a cyclic subdigraph of D is not monochromatic, if such a cycle does exist in D. If $u, v \in V(D)$ & $uv \in A(D)$, then vertices u and v are the same color and in I_{vn}. However, when the arc vu is added to $A(D)$ to create a symmetric arc between u and v, then u and v

[50]Refer to [51].

cannot be the same color. If $u, v, w \in V(D)$ & $uv, vw, uw \in A(D)$, then u, v, w are the same color and in I_{vn} due to D being acyclic. An example is provided with *Figure 3* and *Figure 5*, which are digraphs corresponding to G from *Figure 1*. However, if uw from the previous arc set is changed to wu, then the vertices u, v, w cannot be in the same color class. This occurs because the cycle will connect the An example is provided with *Figure 4* and *Figure 6*. Symmetric arcs between a pair of vertices is a type of cycle. In this paper, a cycle will refer to instances not relating to symmetric arcs.

The *chromatic number of a digraph* $\chi(D)$ is the smallest positive integer n such that there is a proper n-coloring of D. If $d_k(D) = n$, then D is considered to be n-chromatic. If D produces a subdigraph that is a cycle, then the subdigraph cannot be monochromatic and a proper coloring. Thus, D with a cycle will have a higher chromatic number than if D did not contain a cycle. This is demonstrate in *Figures 5* and 6.

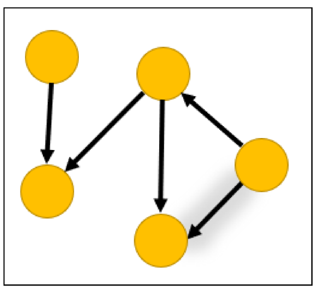


Figure 3
An orientation of $D(G)$ with a cycle and no symmetric arcs
$\chi(D) = 2$

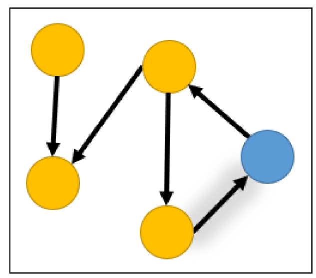


Figure 4
An orientation of $D(G)$ without cycles and no symmetric arcs
$\chi(D) = 1$

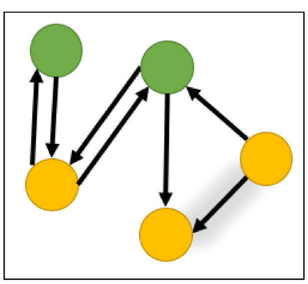


Figure 5
An orientation of $D(G)$ with a cycle and symmetric arcs
$\chi(D) = 3$

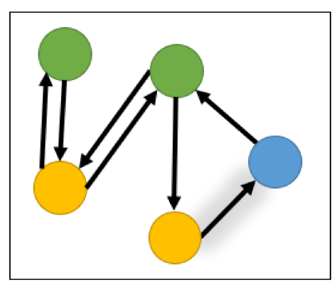


Figure 6
An orientation of $D(G)$ without cycle and symmetric arcs
$\chi(D) = 2$

Dichromatic Number of a Digraph

Definition

The dichromatic number of a digraph is defined as finding the minimum number of colors necessary to color D in such a way that the chromatic classes induce acyclic subdigraphs in D. Neumann-Lara defines the dichromatic number of a digraph as the following.

> An *acyclic n-coloring of a digraph D* is a function $f : V(D) \to I_n$ such that $f^{-1}(i)$ creates an acyclic subdigraph in D for each $i \in I_n$. The **dichromatic number** $d_k(D)$ of D is the minimum n such that there exists an acyclic n-coloring of D. D is said to be *n-dichromatic* if $d_k(D) = n$ and *minimal n-dichromatic* if $d_k(D) = n$ and $d_k(D_0) < n$ for every proper subdigraph D_0 of D.
>
> *Remark 1.* (i) $d_k(D_1) \leqslant d_k(D_2)$ whenever $D_1 \subseteq D_2$.
> (ii) If $d_k(D) \geqslant n$, D contains a minimal n-dichromatic subdigraph.
> (iii) If D is minimal $(n + 1)$-dichromatic and $(u, v) \in A(D)$, then $d_k(D-(u, v)) = n$ and $f(u) = f(v)$ for every acyclic n-coloring of $D-(u, v)$.
>
> *Remark 2.* $d_k(G^*) = \chi(G)$. Moreover, if G is minimal n-chromatic, then G^* is minimal n-dichromatic.

Let's break down the definition and its remarks. An acyclic n-coloring of a digraph contains the function $f : V(D) \to I_n$, which f refers to the vertices of the digraph that are placed in their respective color classes in I_n. The inverse function $f^{-1}(i)$ refers to the group of vertices of one color i from a color class in I_n. This definition of an acyclic n-coloring means that the vertices of one color class will create a subdigraph in D that is acyclic. This will apply to each group of vertices of another color in I_n.

The dichromatic number of D describes the least number of n colors such that D will have an acyclic subdigraph for each color class. A digraph D that has $d_k(D)$ equal to the number of colors used to color D will then be n-dichromatic. D_0 represents the proper subdigraphs of D, proper meaning a proper coloring of the subdigraph. A digraph is minimal n-dichromatic when the subdigraph will have a dichromatic number smaller than the digraph it corresponds to.

Remark 1. (i) A digraph D_1 is a subset of another digraph D_2 when $V(D_1)$ is a subset of $V(D_2)$. When this happens, $d_k(D_1) \leqslant d_k(D_2)$. (ii) This is self-explanatory. (iii) D is minimal $(n + 1)$-dichromatic when $d_k(D) = n + 1$ and $d_k(D_0) < (n + 1)$, where D_0 represents the proper subdigraphs of D. The dichromatic number of D after removing the arc (u, v) will be n. Then, the color of vertices u and v will be the same for every acyclic n-coloring of the digraph without arc uv.

Remark 2. G^* is a digraph that is defined by the following two properties. (i) The vertex set of the digraph $V(G^*)$ is the same vertex set as G such that $V(G^*) = V(G)$. (ii) The arc $uv \in A(G^*)$ can exist iff the edge $[u, v] \in E(G)$), where $[u, v]$ in $E(G)$ is two arcs uv and vu in $A(G^*)$. G^* is similar to $D(G)$, although G^* specifies that there mst be a symmetric pair of arcs for the edges that correspond to G. When the dichromatic number of G^* is found, it will then be the same as chromatic number of G.

Examples of the $d_k(D)$

Example 1 This first example goes through the process of taking digraphs corresponding to a graph and shows the variations that can come out of it.

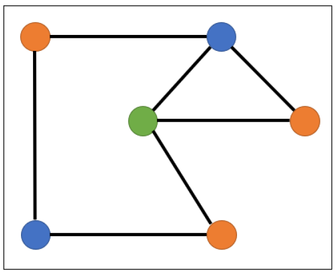


Figure 7
Graph with $\chi(G) = 3$
Start with a graph.

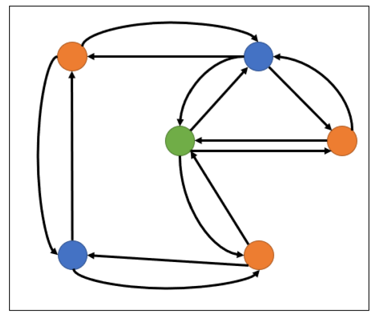


Figure 8
Create a digraph, $D(G)$ or D

In *Figure 7*, a graph G is first constructed for which digraphs will correspond to it. G in particular is shown to have a chromatic number of 3. G could have been colored differently, but a proper k-chromatic coloring is shown. One $D(G)$ is shown in *Figure 8*, which also has a proper coloring. Say $D(G)$ of *Figure 8* is 3-colorable, then $I_3 = \{orange, blue, green\}$. $D(G)$ could have been colored to have n colors be equal to the total number of vertices in $D(G)$, which would be 6. *Figure 11-13* attempts to shows what happens if $D(G)$ is colored differently.

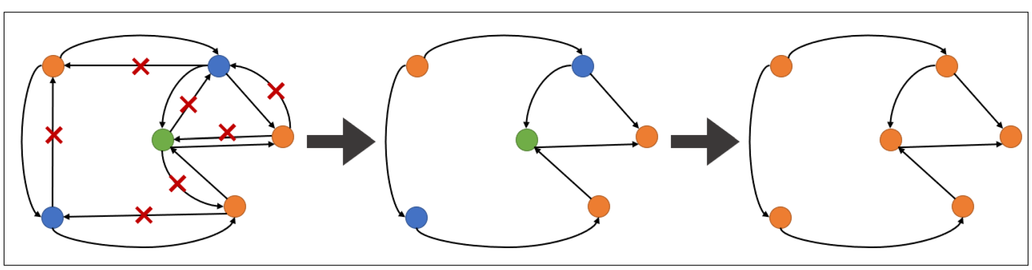


Figure 9
Remove arcs from D in *Figure 8* to create D_0 (graph located on far right) that contains no cycles and is monochromatic when recolored.
This will apply to each color i, in this case also applies to blue and green.

Figure 9 demonstrates the numerous steps to ensure that the D contains an acyclic n-coloring, which it shows that it does in fact contain an acyclic n-coloring. Since D

satisfies the definition of the dichromatic number, $d_k(D) = 3$. D will also satisfy the definition of a minimum n-dichromatic as the different D_0 that could be constructed from *Figure 9* by removing a different set of arcs would also result in $d_k(D_0)$ to be less than $d_k(D)$. (Going through each possible way D_0 can be drawn would take to much space to demonstrate and is left to the reader). If G^* is allowed to be D in this example, notice that $d_k(G^*)$ from *Figure 8* is the same as $\chi(G)$ from *Figure 7*. This exemplifies remark 2.

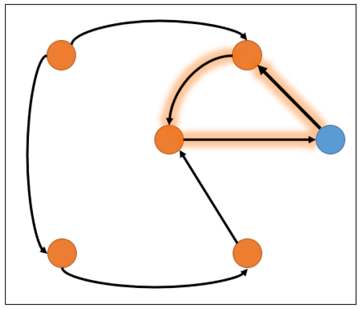


Figure 10
This graph is an example of a subdigraph D_0 that does not fit Neumann's definition.

Figure 11 demonstrates when a proper coloring of a digraph is violated for a D_0 due to a cycle that prevents monochromaticity.

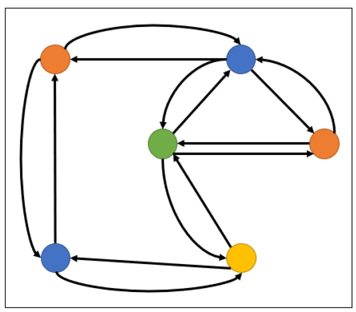


Figure 11
Create a digraph that corresponds to G if G is 4-colorable $\chi(G) = 3$
$d_k(D) = 3$

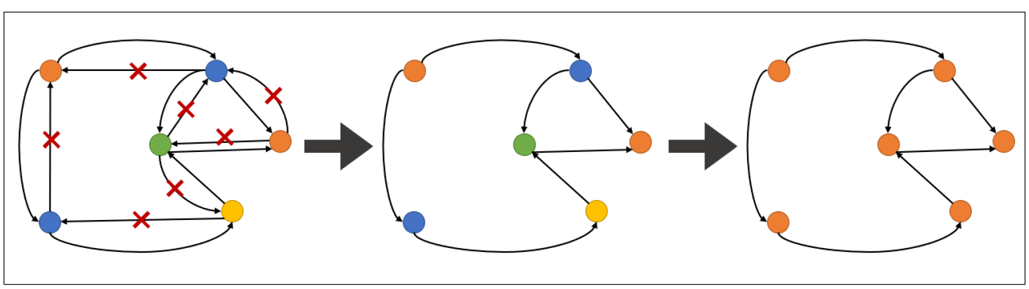


Figure 12
Remove arcs to create a subdigraph D_0 (graph located on far right) that contains no cycles and is monochromatic.

This will apply to each color i, in this case also applies to blue, green, and yellow.

Figures 11 & 12 demonstrate different things when the $D(G)$ contains another coloring. If $D(G)$ is properly colored but not k-chromatic, then a dichromatic number still results the same as in *Figure 8* but will not be minimal n-dichromatic.

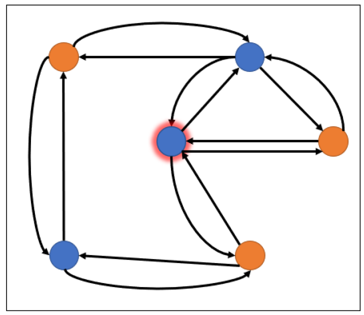


Figure 13
Create a digraph that corresponds to G if $\chi(G) = 2$
$d_k(D)$ ‡ 2 as it will violate the definition of a proper coloring of D.

Figure 13 shows what happens if G from *Figure 7* would have been given a $D(G)$ that was 2-colored. This doesn't work for the clear reason that it doesn't fit the definition of a proper coloring.

<u>Example 2</u> In this example, another graph is created with a corresponding digraph that satisfies the definition of the dichromatic number.

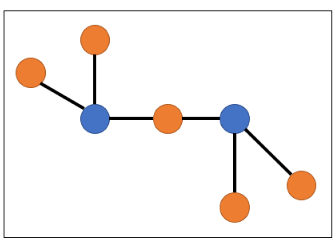


Figure 14
Graph with $\chi(G) = 2$
Start with a G.

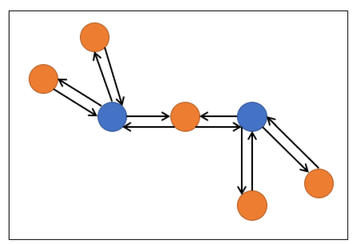


Figure 15
Create a $D(G)$
$d_k(D) = 2$

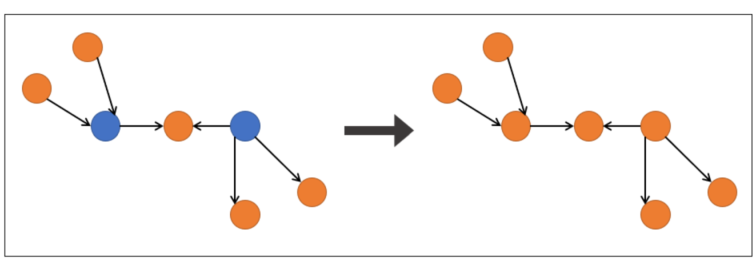


Figure 16
Remove arcs to create a subdigraph D_0 that contains no cycles and is monochromatic. This will apply to each color i.

Example 3 This example shows when G is a K_6.

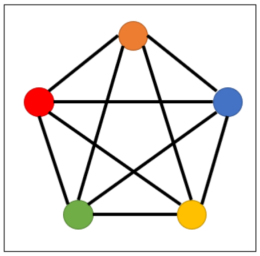


Figure 17
Graph with $\chi(G) = 5$

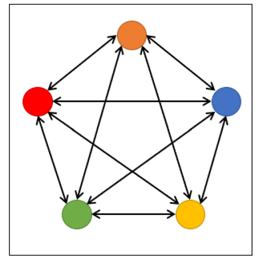


Figure 18
Create a $D(G)$
$d_k(D) = 5$

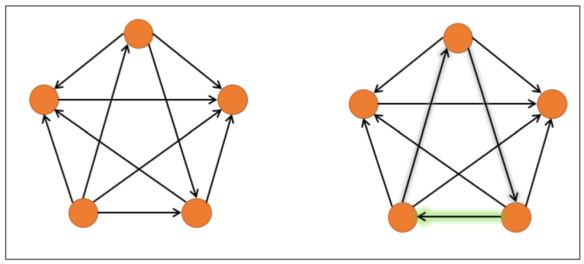


Figure 19
Remove arcs to create D_0. Subdigraph on the right is cyclic and won't be considered.

The dichromatic number of a digraph is a generalization of the chromatic number of a graph. Although the dichromatic number is tedious, it proves to be very useful when describing the chromatic properties of a graph. The concept has been proven to be useful through the numerous articles that have cited Neumann's paper.

Reflection on Víctor Neumann-Lara's Life, 1982-2004

Prior to the 1970's, there was not a diverse set of mathematics to study in México as the two main subjects of study were topology and algebra nor was there much research being produced in México. Víctor had the vision to expand the variety of math taught and researched in México and contributed to a trend of Mexican mathematicians studying mathematics that were not yet available. By the 1980's, UNAM was the only university in México to offer any specific studies and research in mathematics. Víctor used his platform as an established professor and researcher by this time to find new methods of encouraging and recruiting new students. One such method was through the conference "Coloquio Víctor Neumann-Lara de Teoría de las Gráficas, Combinatoria y sus Aplicaciones", which Víctor founded and hosted in 1985. The first conference was held at CIMAT[51]. The conference Víctor established has been held exclusively in

[51] The"Centro de Investigación en Matemáticas" or the Center of Research in Mathematics.

México every year since 1985 and continues to run to this day. It invites mathematicians from all over the world to discuss and share their research in graph theory and combinatorics. He has also participated in numerous conferences held in Mexico and internationally.

His influence on his students and other mathematicians has contributed to the development of graph theory, topology, combinatorics, and applied mathematics. He led the way for mathematical research in México through being a founder of the mathematics institute at UNAM to increase the number of mathematicians and researchers from México. In 1998, he was awarded UNAM's highest award, Premio Universidad Nacional en investigación en Ciencias Exactas[52], for his work. This made him was very happy to be acknowledged for his work.

One of Víctor's passions was teaching, which we can see from his long history as a professor and drive to expand knowledge of mathematics to México at large. With his teaching style, he posed difficult open questions to his graduate students that he advised in hopes of stimulating their creativity and thinking in solving the problem. However, the way he taught required much time for results which caused him to dislike the academic requirements of the institution since it took time away from his students to really invest in the math problems. It's probably from his teaching perspective that he did not bother pursuing the formalities of obtaining a doctorate[53]. When he was teaching classes, he would engage his students through creative approaches by incorporating mythology of sphinxes and sirens into lessons of combinatorics in discrete mathematics. He was always ready to give visual explanations with a bag of rainbow chalk he would carry in his shirt pocket whenever he was asked for help. Víctor also integrated games into his lessons as he enjoyed teaching and solving problems with a simple approach to difficult problems. Neumann-Lara even invented a game called "El Timbiriche Huaxteco"[54]. This game is similar to dots and boxes and requires two players and a graph. Points are won by deleting edges off the graph to isolate an edge. Each player takes turns deleting edges until there are no more moves possible. The player with the most points wins[55].

Neumann taught hundreds of courses in universities around the world. He worked well with his students and published some papers with them. He also influenced his students to pursue their own contributions to graph theory. The following is a small list of students who wrote papers in graph theory: B. Abrego, S. Fernandez-Merchant, M.E. Houle, F. Hurtado, M. Noy, and E. Rivera-Campo, H. Galeana- Sanchez, R. Rojas-Monroy, J.J. Montellano-Ballesteros. It is to no surprise that some of his student went on to pursue mathematical research and become professors. He directed 16

[52]The National University Prize of research in the Exact Sciences. [47]

[53]Dr. Isablea Puga wrote a profile on Víctor, which included this sentiment.

[54]Reference to game in [10][59]

[55]Refer to [9].

undergraduate theses, 1 master's thesis, and 2 doctoral theses.

Víctor was known more than just for his interest in mathematics. He also had interests in politics, history, animals, literature, poetry, the Indigenous people of México, and life in general. He collected many books and created his own personal collection of book ranging from mathematics, to poetry, languages, and much more. In addition to his work in mathematics, Víctor was well-known by those around him for his passion for poetry. He started writing poetry at a young age and was highly influenced by his high school teacher, Pellicer; they became lifelong friends. Over his lifetime, he would continue to write poetry. His poetry group that he met with twice a week had also encouraged him to publish his work. Víctor did publish a book of his work in 1986 titled,*Líneas en el Agua* with illustration by Pellicer. His work was reviewed in the first issue of UNAM's Poetry Paper[56] and was described by Monsalvo that "Víctor Neumann fills the words of plasticity, strength and energy such that they can feel his poetic sense in a magical complex of images"[57].

As said by Clara Grima[58], Víctor was passionate about language and was careful about what word he used to describe graphs in Spanish. There are two words in Spanish for graphs: *grafos* and *gráficas*. Víctor used the term *gráfica* because he felt they were objects that held beauty and should have a feminine name. In addition, he was a lover of life and all it had to offer. He would even say that life was more important than the sciences. Víctor passed away with a community that adored him, his work, and contributions to mathematics in México and the world.

Erdös used to say the ideal way to pass away would be at the end of giving a math talk and leaving an open problem for the next generations of mathematicians to solve.[59] Víctor Neumann-Lara did almost just that when he passed away February 26, 2004 during the 19^{th} annual graph theory and combinatorics conference he had founded. At the conference, he passed away while delivering a talk on a problem he had been trying to solve for years. Although leaving early in the year, 2004 ended up being the year Víctor published the most. A reason to explain this non-stop work he had done is due to previous collaborations he had with other mathematicians. They included Víctor as an author for his contributions to each of their published reviewed papers. In fact, Víctor left an impressive record having 13 papers published as an author after 2004 with his last paper being published in 2013 for a total of 87 publications. His contributions to mathematics has been tremendous and have shaped how mathematicians view graph theory today. México's scientific and mathematical communities have greatly benefited from Víctor's leadership to expand research. I believe he is still overlooked within the general mathematics community despite Víctor's paper "*The Dichromatic number of a*

[56]Refer to [34].
[57]Some of Neumann's poetry is translate and provided in the Appendix [3]
[58]Blog post by Clara and Raquel. [23]
[59]Refer to [56].

Digraph" being cited over a hundred times.

Closing Remarks

This thesis originated Spring of 2019 when the Rocky Mountain section of the MAA introduced the first *History of Math Student Poster Contest.* I had decided to pick a mathematician who was not European, American, nor well known. After some searching, I came across Neumann on a site listing some important Mexican mathematicians. The website only gave brief descriptions for each mathematician causing me to scour the web for more information on Neumann, which proved difficult. I noticed that there is a lack of complete and comprehensive informative sources surrounding historical mathematicians who are Latinx or of indigenous backgrounds.

This is why I wrote this paper to hopefully give a glimpse into the life of mathematician who had a vision of expanding research in a country that was not known for expanding mathematics prior to the 1970's. I wrote this paper also to make a compelling argument in favor of diversifying mathematics everywhere and learning more about the lesser known mathematicians who have not had their full story told. This project also threw me into the midst of what it was like to write an ethnomathematical paper by elevating an unrecognized figure in the mathematics community and show the intersections between math, history, and hopefully culture. I am honored to have written about Víctor Neumann-Lara as I grew to admire his skills, intellect, compassion, and resourcefulness throughout the writing process.

Acknowledgments

I would like to first thank the the Colorado College Mathematics and Computer Science Department for encouraging my studies in mathematics and my pursuit in a historical math thesis. A special thanks to my thesis advisor, Dr. Marlow Anderson, for guiding me through each step and correction. Also, thank you to Dr. Kirsten Hogenson for helping me understand the graph theory.

In addition, I would like to thank Dr. Juan José Montellano (Instiuto de Matemáticas, UNAM), Dr. Eduardo Rivera-Campo (Departamento de Matemáticas, Universidad Autónoma Metropolitana-Iztapalapa), Dr. Isabel Puga (Facultad de Ciencias, UNAM) for providing me important information from their interactions with Víctor Neumann-Lara. Lastly, a very warm welcome to Dr. Max Neumann-Coto, Víctor's son, for providing me with valuable information about Víctor Neumann-Lara I would not have otherwise have had access.

I also want to thank all my friends, family, and co-workers for encouraging me in pursuing this work!

Appendix

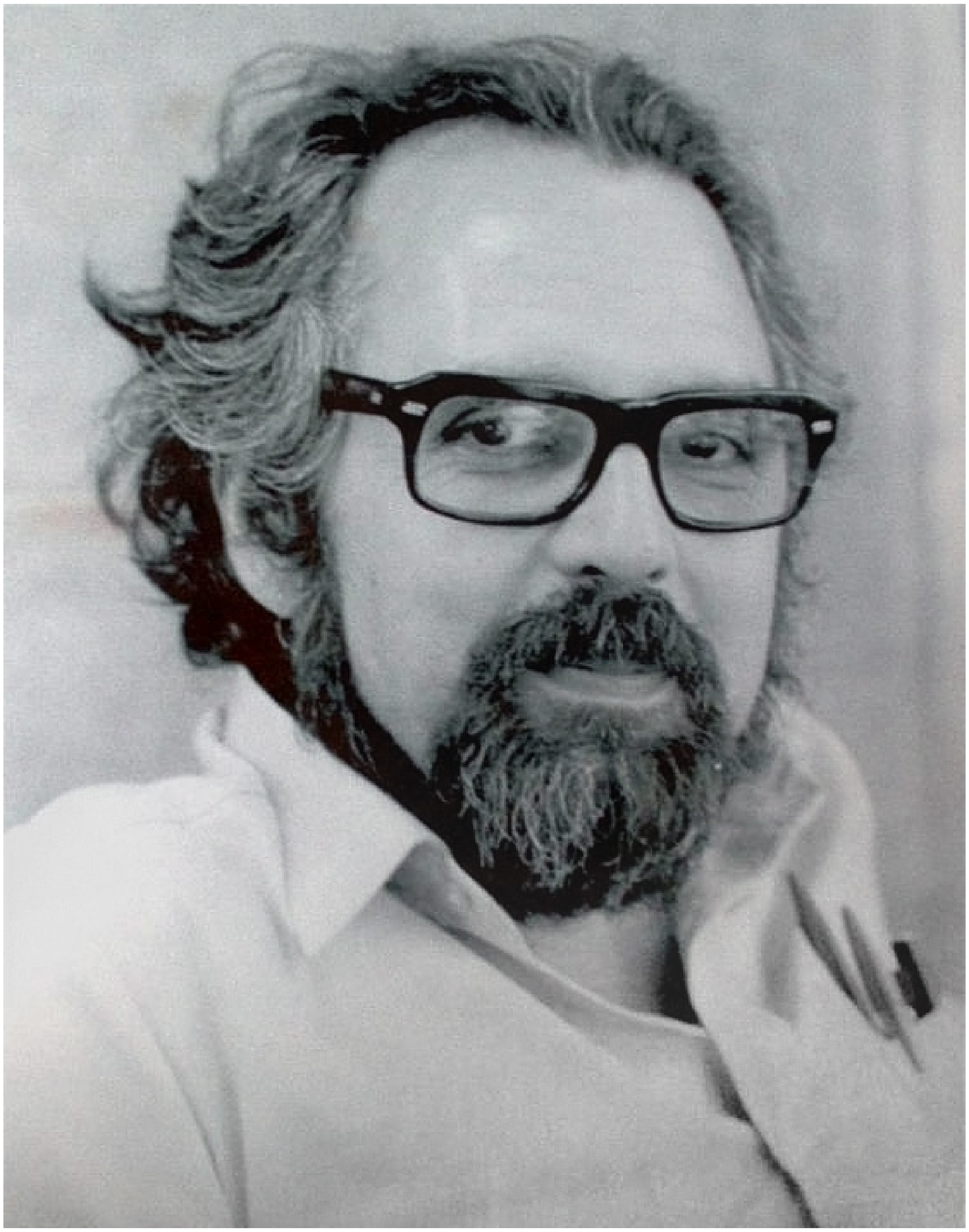
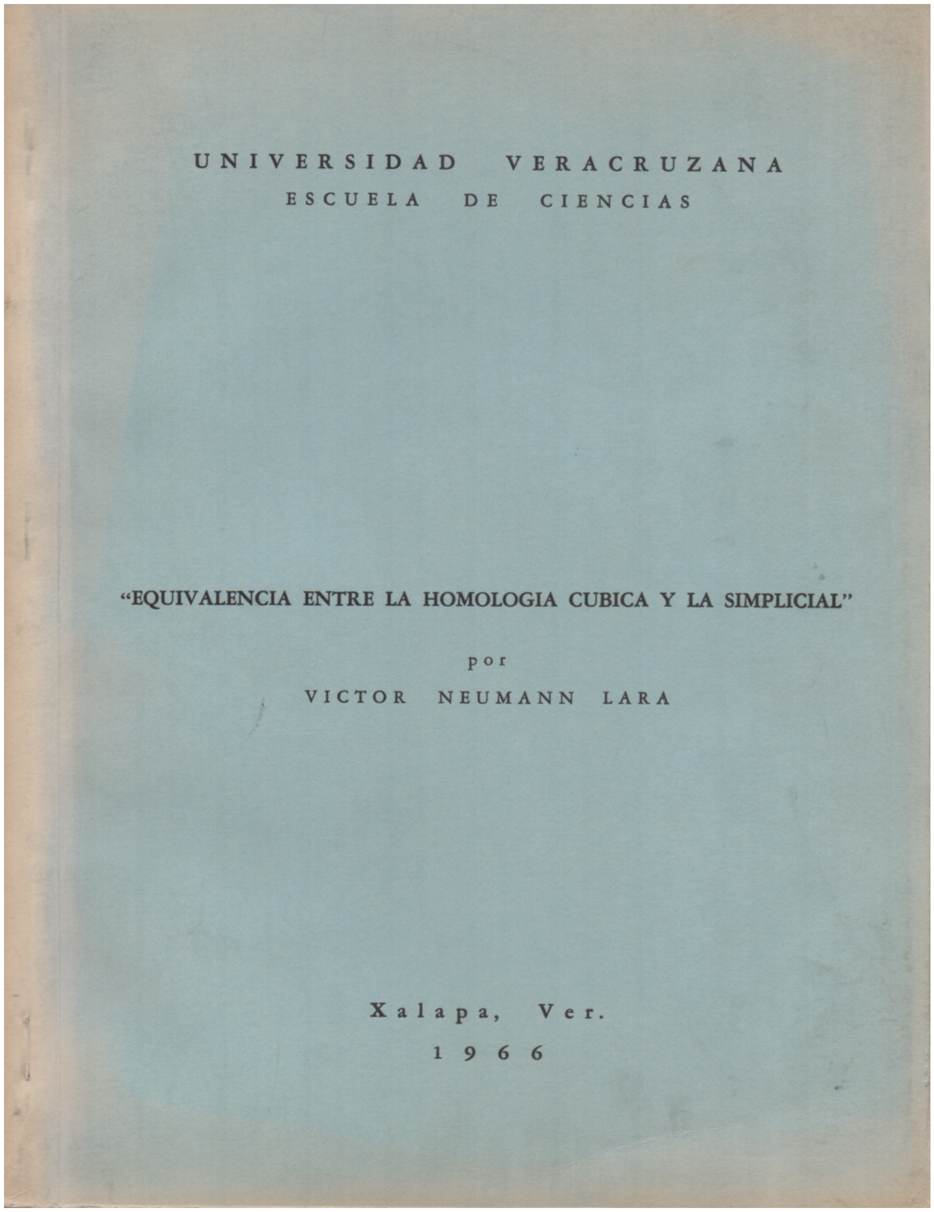
1. 

Photo of Víctor Neumann-Lara

2. 

Victor Neumann-Lara's Undergraduate Thesis Cover
Advisor: Guillermo Torres Published at the Universidad Veracruzana
Photo courtesy of Max Neumann-Coto[38]

3. Spanish with English translations from "Líneas En El Agua" by Víctor Neumann-Lara. Stanzas taken from Sergio Monsalvo article.

- "A partir de esta fecha/ añado al desayuno/ la leche de las primeras constelaciones./ Me desentiendo de la muerte./ Me declaro insolvente ante los años./ Abandono en un parque/ la bufanda de la memoria./ Me tiño las manos de azul/ y alzo a la providencia de tus ojos/ lenguas de girasoles".
  - "From this date / I add to breakfast / the milk of the first constellations./ I

disregard death. / I declare myself insolvent before the years. / I leave in a park / the scarf of memory./ I dye my hands blue / and raise the providence of your eyes / sunflower tongues".

- "Palabras/ en desbandada/ como peces de sombra/ o murciélago/ llegan. desde grutaas antiguas/ se aplastan/como fosiles/ en este trozo de papel./ Dispénsalas/ Que no acampen/ en este abigarrado/ mundo en blanco".
  - "Words / in disarray / like shadow fish / or bats / arrive. from ancient grottos / they are crushed / like fossils / on this piece of paper./ Dispense them / Let them not camp / in this motley / blank world ".
- "De toda esa confusa/ constelación/ en que tus suenos/ amanecen y se derrumban/ del incesante flugo de olas muertas/ rescatas cadaveres de palabras/ y cangrejos rotos./ Acaso unoscuantos guijarros blancos/hundidos en la arena".
  - "From all that confused / constellation / in which your dreams / dawn and collapse / from the incessant flow of dead waves / rescue corpses of words / and broken crabs. / Perhaps a few white pebbles / sunk in the sand ".
- "Como un fruto de sombr/ caigo del sueño/ al día./ En juego/ cuyas reglas ignoro/ alguien ha puesto en mi café/ arena/ de lunas y peces./ ¿Qué telegrafía hará la relación/ de esta confusa vacuidad?/ Signos/ desplegados/ como banderas en las vasta/ arboladura del vacío./ Hoy es la rosa ausente".
  - "Like a fruit of the shadow / I fall from sleep / day. / At stake / whose rules I do not know / someone has put in my coffee / sand / of moons and fish / What telegraphy will the relationship / of this confused emptiness? / Signs / displayed / like flags in the vast / hoist of the void / Today is the absent rose "
- "Mientras la tarde/ se borra el rostro con los dedos/ y los fantasmas de las flores/ foforecen en la penumbra/ de las piedras/ desde muy lejos llegan voces/ estrañas/ desdoblando ventanas que se trizan/ con un clamor de hierba./ La noche invade el centro de las cosas./ En el fondo/ de la catástrfe incumplida/ ocultos ríos se despeñan. Alzan/ el verde limo de los ahogados./ Los cabellos violetas resplandecen./ Son líneas en el agua."
  - "While the evening / erases the face with his fingers / and ghosts of the flowers / they offer in the gloom / of the stones / from far away come voices / strange / unfolding windows that are broken / with a clamor of grass. / The night invades the center of things. / In the background / of the unfulfilled catastrophe / hidden rivers rise. They rise / the green slime of the drowned. / Violet hair glows. / They are lines in the water. "

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